

Partial spontaneous breaking of two-dimensional supersymmetry

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Abstract

We construct low-energy Goldstone superfield actions describing various patterns of the partial spontaneous breakdown of two-dimensional $N = (1, 1)$, $N = (2, 0)$ and $N = (2, 2)$ supersymmetries, with the main focus on the last case. These nonlinear actions admit a representation in the superspace of the unbroken supersymmetry as well as in a superspace of the full supersymmetry. The natural setup for implementing the partial breaking in a self-consistent way is provided by the appropriate central extensions of $D = 2$ supersymmetries, with the central charges generating shift symmetries on the Goldstone superfields. The Goldstone superfield actions can be interpreted as manifestly world-sheet supersymmetric actions in the static gauge of some superstrings and D1-branes in $D = 3$ and $D = 4$ Minkowski spaces. As an essentially new example, we elaborate on the action representing the 1/4 partial breaking pattern $N = (2, 2) \rightarrow N = (1, 0)$.

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1 Introduction

Supersymmetric models in two dimensions, in particular $N=(2,2)$ ones, were a subject of numerous studies (see, e.g., [1]-[6]). The generic source of interest in such models is their tight relation to strings, integrable systems and $D = 2$ conformal theory. In particular, $D = 2$ conformal field theories with $N = (2,2)$ supersymmetry are considered as candidate vacua for perturbative string theory. More specifically, $N = (2,2)$ models, being invariant under the simplest extended supersymmetry in $D = 2$ and allowing for concise off-shell superfield formulations [1]-[4], provide the proper laboratory for analyzing the characteristic features they share with more complicated higher-dimensional superfield theories.

One of such generic features is the phenomenon of spontaneous partial breaking of global supersymmetry (PBGs) [7]-[16]. The first self-consistent example of the Goldstone-fermion $D = 2$ model with the spontaneous partial breaking of global supersymmetry was constructed in ref. [8]. There, the partial breaking $N = (2,2) \rightarrow N = (2,0)$ was triggered by a topologically non-trivial BPS classical solution preserving one half of the original supersymmetry, the one generated by two left supercharges only. The full $N = (2,2)$ supersymmetry algebra was found to include two central charges (also spontaneously broken), and the resulting invariant action proved to be the static gauge form of the Green-Schwarz action of $N = 1, D = 4$ superstring. The construction of ref. [8] essentially exploited the methods of nonlinear realizations of global supersymmetry. Recently, these methods were applied to study various PBGS options in the models with $D=4$, $N=2$ [10]-[13], $D=4$, $N=4$ (or $D=10$, $N = 1$) [14] and $D=3$, $N=2$ [15] supersymmetries, and to reveal their relationships with branes. The interplay between the PBGS description of branes and the one based on the superembedding approach ([17] and references therein) was studied in [18], [19].

There still remain some problems with the treatment of the PBGS phenomenon which may be clarified by further studying it within $D = 2$ models. In particular, this regards the linear realizations of PBGS and their relation to the universal nonlinear realizations description. Also, besides the PBGS option leading to the $N = 1, D = 4$ superstring along the line of ref. [8], there exist others which have not been properly investigated until now.

The basic aim of the present paper is to fill this gap. We construct the manifestly worldvolume supersymmetric Goldstone superfield actions for the PBGS patterns $N = (2,2) \rightarrow N = (1,1)$ and $N = (2,2) \rightarrow N = (1,0)$ and give them an interpretation as the static-gauge actions of some superstrings and D1-branes. We also reproduce, in a new setting, the $N = (2,2) \rightarrow N = (2,0)$ model of ref. [8]. As a by-product, we find some interesting examples of PBGS in the cases of $N = (1,1)$ and $N = (2,0)$ supersymmetries. All the Goldstone superfield actions are written in a two-fold way: as integrals over superspaces of unbroken supersymmetry and as integrals over superspaces of the full spontaneously broken supersymmetry, which is manifest in the latter formulation. This allows us to unveil the relationship between linear and nonlinear realizations of the partial breaking patterns considered. For self-consistency, in all non-trivial cases it proves necessary to proceed from the properly central-charge extended $D = 2$ supersymmetry. These central charges generate shift symmetries of the Goldstone superfields and play a crucial role both in the implementation of the partial breaking and in the superbrane interpretation of the Goldstone superfield actions.

In our study, we systematically make use of the general relationship between linear and nonlinear realizations of supersymmetry [20]. It was already applied to the PBGS case in a recent paper [21].

2 ABC of $N=(2,2)$, $D=2$ superspace

The aim of this Section is to adduce the necessary information about the superspaces of $N = (2, 2)$, $D = 2$ supersymmetry which is the basic object of our study in the present paper. We also focus on the structure of central-charge extensions of this supersymmetry.

The full $D=2$, $N = (2, 2)$ superspace $z = (z_l, z_r)$ consists of two light-cone sectors, the left $(2,0)$ and right $(0,2)$ ones z_l and z_r , parametrized, respectively, by the coordinates

$$z_l = (x^\pm, \theta^\pm, \bar{\theta}^\pm) , \quad z_r = (x^\mp, \theta^\mp, \bar{\theta}^\mp) \quad (2.1)$$

with the $SO(1,1)$ weights $(\pm 2, \pm 1, \pm 1)$. Sometimes it is convenient to parametrize $N = (2, 2)$ superspace by four real Grassmann coordinates $\theta_1^\pm, \theta_2^\pm$,

$$\begin{aligned} \theta_1^\pm &= \frac{1}{\sqrt{2}}(\theta^\pm + \bar{\theta}^\pm) , & \theta_2^\pm &= \frac{i}{\sqrt{2}}(\bar{\theta}^\pm - \theta^\pm) , \\ \theta^\pm &= \frac{1}{\sqrt{2}}(\theta_1^\pm + i\theta_2^\pm) , & \bar{\theta}^\pm &= \frac{1}{\sqrt{2}}(\theta_1^\pm - i\theta_2^\pm) . \end{aligned} \quad (2.2)$$

The algebra of spinor derivatives in the $(2,0)$ and $(0,2)$ sectors has the following form:

$$\{D_+, \bar{D}_+\} = 2i\partial_+ \equiv 2P_+ , \quad D_+ D_+ = 0 , \quad \bar{D}_+ \bar{D}_+ = 0 , \quad (2.3)$$

$$\{D_-, \bar{D}_-\} = 2i\partial_- \equiv 2P_- , \quad D_- D_- = 0 , \quad \bar{D}_- \bar{D}_- = 0 . \quad (2.4)$$

The crossing relations admit four real $SO(1,1)$ singlet central charges

$$\begin{aligned} \{D_+, \bar{D}_-\} &= 2(Z_1 + iZ_2) , & \{\bar{D}_+, D_-\} &= 2(Z_1 - iZ_2) , \\ \{D_+, D_-\} &= 2(Z_3 + iZ_4) , & \{\bar{D}_+, \bar{D}_-\} &= 2(Z_3 - iZ_4) . \end{aligned} \quad (2.5)$$

These spinor covariant derivatives can be chosen in the following explicit form:

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+ \partial_+ - \bar{\theta}^- (Z_1 + iZ_2) + \theta^- (Z_3 + iZ_4) , \\ \bar{D}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+ \partial_+ - \bar{\theta}^- (Z_3 - iZ_4) + \theta^- (Z_1 - iZ_2) , \\ D_- &= \frac{\partial}{\partial\theta^-} - i\bar{\theta}^- \partial_- - \bar{\theta}^+ (Z_1 - iZ_2) + \theta^+ (Z_3 + iZ_4) , \\ \bar{D}_- &= -\frac{\partial}{\partial\bar{\theta}^-} + i\theta^- \partial_- - \bar{\theta}^+ (Z_3 - iZ_4) + \theta^+ (Z_1 + iZ_2) . \end{aligned} \quad (2.6)$$

Note that this definition is not unique in view of possibility to perform similarity transformations

$$(D, \bar{D}) \Rightarrow (\tilde{D}, \tilde{\bar{D}}) = e^{-A}(D, \bar{D})e^A , \quad (2.7)$$

where the operator A is a linear combination of central charges with the coefficients bilinear in Grassmann coordinates (all differential operators, e.g., generators of supersymmetry, and $N = (2, 2)$ superfields should be simultaneously transformed). For instance, making such a transformation with

$$A = -\theta^+ \theta^- (Z_3 + iZ_4) - \bar{\theta}^+ \bar{\theta}^- (Z_3 - iZ_4) + \bar{\theta}^+ \theta^- (Z_1 - iZ_2) + \theta^+ \bar{\theta}^- (Z_1 + iZ_2) , \quad (2.8)$$

one can entirely remove the central-charge terms from D_+, \bar{D}_+ at cost of appearance of the factor 2 in front of such terms in D_-, \bar{D}_- . In this particular case $A = -A^\dagger$, so the transformation preserves the simple properties of covariant derivatives under complex conjugation. In other cases the conjugation rules become more complicated.

For the spinor derivatives in the real parametrization (2.2)

$$D_{\pm}^1 = \frac{1}{\sqrt{2}}(D_{\pm} - \bar{D}_{\pm}) , \quad D_{\pm}^2 = \frac{i}{\sqrt{2}}(D_{\pm} + \bar{D}_{\pm}) , \quad (2.9)$$

eqs. (2.3) - (2.5) imply the following relations:

$$D_+^1 D_+^1 = D_+^2 D_+^2 = -i\partial_{\mp} , \quad D_-^1 D_-^1 = D_-^2 D_-^2 = -i\partial_{=} , \quad (2.10)$$

$$\{D_+^1, D_+^2\} = 0 , \quad \{D_-^1, D_-^2\} = 0 . \quad (2.11)$$

$$\{D_+^1, D_-^1\} = 2(Z_3 - Z_1) , \quad \{D_+^2, D_-^2\} = -2(Z_1 + Z_3) , \quad (2.12)$$

$$\{D_+^1, D_-^2\} = -2(Z_2 + Z_4) , \quad \{D_+^2, D_-^1\} = 2(Z_2 - Z_4) . \quad (2.13)$$

The explicit form of these derivatives can be easily found from (2.6) and (2.2), for instance

$$D_+^1 = \frac{\partial}{\partial\theta_+^1} - i\theta_+^1 \partial_{\mp} - \theta_+^1 (Z_1 - Z_3) - \theta_+^2 (Z_2 + Z_4) . \quad (2.14)$$

We shall use the following notation for the infinitesimal $N = (2, 2)$ supersymmetry transformation of some $N = (2, 2)$ superfield $\phi(z)$:

$$\begin{aligned} \delta\phi &= (\epsilon Q)\phi, \\ (\epsilon Q) &= \epsilon^+ Q_+ + \bar{\epsilon}^+ \bar{Q}_+ + \epsilon^- Q_- + \bar{\epsilon}^- \bar{Q}_- = \epsilon_k^+ Q_+^k + \epsilon_k^- Q_-^k = -(\epsilon Q)^{\dagger} \end{aligned} \quad (2.15)$$

and the corresponding finite transformation is

$$\phi'(z) \equiv \phi(\epsilon, z) = \exp\{\epsilon Q\}\phi(z) . \quad (2.16)$$

Here $k = 1, 2$ and

$$Q_{\pm}^1 = \frac{1}{\sqrt{2}}(Q_{\pm} + \bar{Q}_{\pm}) , \quad Q_{\pm}^2 = \frac{i}{\sqrt{2}}(Q_{\pm} - \bar{Q}_{\pm}) .$$

With this convention, the algebra of $N = (2, 2)$ supersymmetry has the form:

$$\begin{aligned} \{Q_+, \bar{Q}_+\} &= 2P_{\mp} , \quad \{Q_-, \bar{Q}_-\} = 2P_{=} , \\ \{Q_+, \bar{Q}_-\} &= 2(Z_1 + iZ_2) , \quad \{Q_+, Q_-\} = 2(Z_3 + iZ_4) . \end{aligned} \quad (2.17)$$

The algebra of generators in the real parametrization (2.2), i.e. that of Q_{\pm}^1, Q_{\pm}^2 , can be easily read off from these relations, e.g.,

$$Q_+^1 Q_+^1 = i\partial_{\mp} , \quad Q_-^1 Q_-^1 = i\partial_{=} , \quad \{Q_+^1, Q_-^1\} = 2(Z_1 + Z_3) . \quad (2.18)$$

In what follows we shall deal with the $N = (2, 2)$ superspace without special coordinates for the central charges.¹ These charges will show up only as generators of some shifting isometries realized on fields, i.e. as internal symmetry generators. Thus we shall basically use (in particular, in the rest

¹Adding two such coordinates would actually mean passing to $N = 1, D = 4$ superspace: $N = (2, 2)$ superalgebra with one complex central charge in the crossing anticommutators (it can be $Z_1 \pm iZ_2$ or $Z_3 \pm iZ_4$, these choices are equivalent up to the twist $D_- \leftrightarrow \bar{D}_-$) is isomorphic to the standard $N = 1, D = 4$ Poincaré superalgebra (total of 4 bosonic generators). On the other hand, the situation with *two* independent complex central charges in (2.5) (total of 6 bosonic generators) corresponds to some non-trivial truncation of the extension of $N = 1, D = 4$ Poincaré superalgebra by complex tensorial central charges [22]-[24]. The full such extension (10 bosonic generators) in the $D = 2$ language would correspond to inserting two independent complex charges with the appropriate $SO(1, 1)$ weights also into the r.h.s. of $D_{\pm} D_{\pm}$. Though such further modification of $N = (2, 2)$ superalgebra can certainly have implications in the PBGS and brane contexts (see, e.g., [21] for the case of superparticle), for simplicity we shall not dwell on this possibility here.

of this Section) the spinor derivatives (2.6), (2.14) with the central charge terms neglected. The corresponding superalgebra and superspace reveal the automorphism $U(1)_+ \times U(1)_-$ symmetry (or R -symmetry) realized as independent phase transformations of the left and right pairs of the spinor coordinates

$$(\theta^+, \bar{\theta}^+) , \quad (\theta^-, \bar{\theta}^-) . \quad (2.19)$$

Actually, this symmetry is explicitly broken, at least down to the diagonal $U(1)$, in most $N = (2, 2)$ models.² The integration measure in the full $N = (2, 2)$ superspace is defined by

$$d^6 z = \overline{(d^6 z)} = d^2 x D_- D_+ \bar{D}_+ \bar{D}_- = d^2 x D_+^1 D_-^1 D_+^2 D_-^2 . \quad (2.20)$$

In both light-cone sectors of $N = (2, 2)$ superspace there exist mutually conjugated left and right chiral (chiral and antichiral) subspaces. E.g., in the $(2, 0)$ sector the corresponding coordinate sets are as follows:

$$\zeta_L^{(+)} = (x_L^\pm, \theta^+) , \quad x_L^\pm = x^\pm - i\theta^+ \bar{\theta}^+ = x^\pm - \theta_1^+ \theta_2^+ , \quad (2.21)$$

$$\zeta_R^{(+)} = (x_R^\pm, \bar{\theta}^+) , \quad x_R^\pm = x^\pm + i\theta^+ \bar{\theta}^+ = x^\pm + \theta_1^+ \theta_2^+ . \quad (2.22)$$

The chiral and antichiral coordinates in the $(0, 2)$ sector $\zeta_L^{(-)} = (x_L^\pm, \theta^-)$ and $\zeta_R^{(-)} = (x_R^\pm, \bar{\theta}^-)$ can be introduced analogously. The integration measures in the chiral and antichiral $N = (2, 2)$ superspaces $\zeta_L = (\zeta_L^{(+)}, \zeta_L^{(-)})$ and $\zeta_R = (\zeta_R^{(+)}, \zeta_R^{(-)})$ are defined as

$$d^4 \zeta_L = d^2 \zeta_L^{(-)} d^2 \zeta_L^{(+)} = dx_L^\pm dx_L^\mp D_- D_+ , \quad d^4 \zeta_R = d^2 \zeta_R^{(+)} d^2 \zeta_R^{(-)} = dx_R^\pm dx_R^\mp \bar{D}_+ \bar{D}_- . \quad (2.23)$$

The chiral $N = (2, 2)$ superfields are defined by the following constraints

$$\bar{D}_+ \phi = \bar{D}_- \phi = 0 , \quad (2.24)$$

while the twisted-chiral ones [1, 25] by

$$D_+ \lambda = \bar{D}_- \lambda = 0 . \quad (2.25)$$

The defining constraints for the antichiral and twisted anti-chiral superfields follow from (2.24), (2.25) by complex conjugation. Note that in the case of non-vanishing central charges the constraints (2.24), (2.25) require, respectively,

$$(Z_3 - iZ_4)\phi = 0 \quad \text{and} \quad (Z_3 + iZ_4)\bar{\phi} = 0 , \quad (2.26)$$

or

$$(Z_1 + iZ_2)\lambda = 0 \quad \text{and} \quad (Z_1 - iZ_2)\bar{\lambda} = 0 . \quad (2.27)$$

as their integrability conditions. If these conditions are satisfied, the remaining central-charge terms can be removed from the spinor covariant derivatives in (2.24) or (2.25) by a proper transformation like (2.7). As the result, in such a frame chirality or twisted chirality become manifest despite the presence of central charges in the $N = (2, 2)$ superalgebra. For instance, if some superfield ϕ satisfies the conditions (2.24) with \bar{D}_\pm defined by (2.6), the superfield

$$\tilde{\phi} = e^{-A} \phi , \quad A = \bar{\theta}^+ \theta^- (Z_1 - iZ_2) - \theta^+ \bar{\theta}^- (Z_1 + iZ_2)$$

satisfies the standard chirality conditions with “short” covariant derivatives (containing no central-charge terms), and so it lives on the chiral superspace $\zeta_L = (\zeta_L^{(+)}, \zeta_L^{(-)})$.

²The extended superalgebra (2.5) is still covariant under these automorphisms, provided one ascribes appropriate transformation properties to the complex central charges.

In what follows we shall need some facts about $N = (1, 1)$, $N = (2, 0)$ and $N = (1, 0)$ superspaces as subspaces of the $N = (2, 2)$ one. All the covariant spinor derivatives below are assumed to be “short”, i.e. containing no central-charge terms.

The $N = (1, 1)$ superspace is parametrized by the real spinor coordinates θ_1^+, θ_1^- related to the $N = (2, 2)$ ones according to (2.2). The simplest free $N = (1, 1)$ scalar multiplet model is described by the unconstrained real scalar superfield $\pi(x, \theta_1^+, \theta_1^-)$ with the action

$$S = \frac{1}{2i} \int d^4z D_+^1 \pi D_-^1 \pi \quad (2.28)$$

where

$$d^4z = \overline{(d^4z)} \equiv d^2x iD_+^1 D_-^1 \quad (2.29)$$

is the $N = (1, 1)$ superspace integration measure.

The $N = (2, 0)$ and $N = (0, 2)$ subspaces (not to be confused with the $(2, 0)$ and $(0, 2)$ light-cone sectors) are spanned by the following coordinate sets

$$z^{(+)} = (x^-, z_l) = (x^-, x^\pm, \theta^+, \bar{\theta}^+) , \quad z^{(-)} = (x^\pm, z_r) = (x^\pm, x^-, \theta^-, \bar{\theta}^-) . \quad (2.30)$$

The appropriate integration measures have non-trivial $SO(1, 1)$ weights (∓ 2):

$$d^4z^{(+)} = d^2x D_+ \bar{D}_+ = i d^2x D_+^2 D_+^1 , \quad d^4z^{(-)} = d^2x D_- \bar{D}_- = i d^2x D_-^2 D_-^1 . \quad (2.31)$$

The simplest off-shell representations are comprised by the $N = (2, 0)$ and $N = (0, 2)$ chiral superfields $\varphi(x_L^\pm, x^\pm, \theta^+)$ and $\omega(x_L^\pm, x^\pm, \theta^-)$ with the following free actions

$$S_0(\varphi) \sim \frac{1}{2i} \int d^4z^{(+)} \bar{\varphi} \partial_- \varphi , \quad S_0(\omega) \sim \frac{1}{2i} \int d^4z^{(-)} \bar{\omega} \partial_+ \omega . \quad (2.32)$$

At last, the free action of an unconstrained scalar $N = (1, 0)$ superfield $v(x^\pm, x^\pm, \theta_1^+)$ has the following form:

$$\frac{1}{2} \int d^3z^{(+)} D_+^1 v \partial_- v , \quad d^3z^{(+)} \equiv d^2x iD_+^1 . \quad (2.33)$$

Its $N = (0, 1)$ counterpart is constructed in a similar and evident way.

3 Toy examples

To clarify the basic features of our approach we start with the simplest examples of partial breaking of the $D = 2$ global supersymmetry with two supercharges. In $D = 2$ there exist two different patterns for such breaking, viz. $N = (1, 1) \rightarrow N = (1, 0)$ and $N = (2, 0) \rightarrow N = (1, 0)$.

3.1 $N = (1, 1) \rightarrow N = (1, 0)$

To describe this PBGS option, we should construct $N = (1, 0)$ superfield action possessing one additional spontaneously broken $N = (0, 1)$ supersymmetry. The $N = (1, 0)$ superfield formulation is preferable because only the $N = (1, 0)$ supersymmetry is supposed to remain unbroken and so manifest. As usual, the partial breaking implies the presence of the Goldstone fermion among the component fields of our theory. The simplest possibility is to start with a real bosonic scalar $N = (1, 0)$ superfield $u(x^\pm, x^\pm, \theta^+)$ and to define the real fermionic superfield $\xi^-(x^\pm, x^\pm, \theta^+)$ ³

$$\xi^- \equiv iD_+ u , \quad (3.1)$$

³The real $N = (1, 1)$ superspace coordinates θ^\pm used in this Subsection should not be confused with the complex $N = (2, 2)$ superspace ones.

with

$$D_+ = \frac{\partial}{\partial \theta^+} - i\theta^+ \partial_+, \quad D_+ D_+ = -i\partial_+,$$

whose first component is assumed to be the Goldstone fermion. Since the crucial characteristic feature of the latter is the pure shift in the transformation rule under spontaneously broken $N = (0, 1)$ supersymmetry,

$$\delta \xi^- = \epsilon^- + \dots,$$

the appropriate transformation of u should also contain an inhomogeneous term

$$\delta u = i\epsilon^- \theta^+ + \dots$$

Here, ϵ^- is the transformation parameter

In order to have a linear off-shell realization of this extra $N = (0, 1)$ supersymmetry, we have to add one more fermionic superfield $\eta^+(x^\pm, x^=, \theta^+)$. It is easy to find that the following transformation laws of u and η just constitute the desired $N = (0, 1)$ supersymmetry algebra

$$\begin{aligned} \delta u &= i\epsilon^- \theta^+ + i\epsilon^- \eta^+ \Rightarrow \delta \xi^- = \epsilon^- + \epsilon^- D_+ \eta^+, \\ \delta \eta^+ &= -\epsilon^- \partial_- u. \end{aligned} \quad (3.2)$$

Together with the manifest $N = (1, 0)$ supersymmetry it forms the full $N = (1, 1)$, $D = 2$ supersymmetry.

It is easy to check that the closure of the manifest $N = (1, 0)$ and hidden $N = (0, 1)$ supersymmetries on the superfield u yields a constant shift of u . So in the present case we are facing the central-charge extended $N = (1, 1)$, $D = 2$ supersymmetry algebra:

$$\{\hat{Q}_+, \hat{Q}_-\} = 2Z, \quad \hat{Q}_+ \hat{Q}_+ = P_+ = i\partial_+, \quad \hat{Q}_- \hat{Q}_- = P_- = i\partial_-, \quad (3.3)$$

where Z acts as a pure translation of u :

$$\delta_z u = -2i a[Z, u] = a, \quad a^\dagger = a, \quad (3.4)$$

and “hat” was introduced to distinguish this algebra from the one without central charge.

It is instructive to see how this $N = (1, 0)$ superfield representation could equivalently be deduced from the $N = (1, 1)$ superspace formalism.

To this end, let us consider a scalar real $N = (1, 1)$ superfield $\Phi(x, \theta^+, \theta^-)$ ($[\Phi] = -1$),

$$\Phi(x, \theta^+, \theta^-) = u(x, \theta^+) + i\theta^- \eta^+(x, \theta^+), \quad (3.5)$$

such that it possesses non-trivial transformation properties under the central charge Z which generates pure shifts of Φ (cf. (3.4)):

$$\delta_z \Phi = -2i a[Z, \Phi] = a \Rightarrow Z\Phi = \frac{i}{2}. \quad (3.6)$$

For the moment, the coefficients in the θ^- expansion (3.5) of Φ are arbitrary and not to be identified with the previous $N = (1, 0)$ superfields. The numerical coefficient in the definition of Z in (3.6) was chosen for further convenience: it is defined up to an arbitrary rescaling of Φ .

The generators of the central-charge extended $N = (1, 1)$ superalgebra (3.3) can be chosen so as

$$\begin{aligned} \hat{Q}_+ &= \frac{\partial}{\partial \theta^+} + i\theta^+ \partial_+ = Q_+, \\ \hat{Q}_- &= \frac{\partial}{\partial \theta^-} + i\theta^- \partial_- + 2\theta^+ Z = Q_- + 2\theta^+ Z. \end{aligned} \quad (3.7)$$

This choice is convenient in that Φ has the standard superfield transformation law under $N = (1, 0)$ supersymmetry and, hence, the coefficients in its θ^- expansion (u and η^+) are automatically standard $N = (1, 0)$ superfields⁴. At the same time, Φ , with taking into account (3.6), inhomogeneously transforms under the second supersymmetry:

$$\delta\Phi = \epsilon^- \hat{Q}_- \Phi = i\epsilon^- \theta^+ + \epsilon^- Q_- \Phi. \quad (3.8)$$

It is straightforward to see that for the $N = (1, 0)$ components of Φ (3.5) this law produces just the transformation laws (3.2). Hence, these components can be identified with the $N = (1, 0)$ superfields introduced earlier.

To construct invariants, we need to define the $N = (1, 1)$ spinor covariant derivatives for this special case. These objects, anticommuting with the generators (3.7), are easily seen to be as follows

$$\begin{aligned} \hat{D}_+ &= \frac{\partial}{\partial\theta^+} - i\theta^+ \partial_+ - 2\theta^- Z \equiv D_+ - 2\theta^- Z, \\ \hat{D}_- &= \frac{\partial}{\partial\theta^-} - i\theta^- \partial_- = D_- , \quad D_+ D_+ = -i\partial_+ , \quad D_- D_- = -i\partial_- . \end{aligned} \quad (3.9)$$

So, for constructing $N = (1, 1)$ invariants, we have two covariant quantities:

$$\begin{aligned} \hat{D}_- \Phi &= D_- \Phi = i(\eta^+ - \theta^- \partial_- u) , \\ \hat{D}_+ \Phi &= D_+ \Phi - i\theta^- = D_+ u - i\theta^- (1 + D_+ \eta^+) . \end{aligned} \quad (3.10)$$

Then, like in the case without central charges, eq. (2.28), the simplest invariant action is as follows

$$S_\Phi = \frac{1}{2if^2} \int d^4z \hat{D}_+ \Phi D_- \Phi = \frac{1}{2if^2} \int d^4z D_+ \Phi D_- \Phi - \frac{1}{2f^2} \int d^4z \Phi \equiv S_1 - S_2 , \quad (3.11)$$

where we made use of (3.10) and integrated by parts with respect to D_- (f is a normalization factor of dimension -1). It is easy to see that S_2 is invariant on its own because the inhomogeneous term in the transformation law (3.8) of Φ does not contribute by the definition of the $N = (1, 1)$ integration measure (2.29). So, S_1 is also invariant (up to a shift of the Lagrangian density by a full derivative). In terms of the $N = (1, 0)$ superfields u, η^+ these invariants read

$$\begin{aligned} S_1 &= \frac{1}{2f^2} \int d^3z^{(+)} (D_+ u \partial_- u - i\eta^+ D_+ \eta^+) \\ &= \frac{i}{2f^2} \int d^2x D_+ (D_+ u \partial_- u - i\eta^+ D_+ \eta^+) , \end{aligned} \quad (3.12)$$

$$S_2 = \frac{i}{2f^2} \int d^3z^{(+)} \eta^+ = -\frac{1}{2f^2} \int d^2x D_+ \eta^+ , \quad (3.13)$$

where the $N = (1, 0)$ superspace integration measure is defined in (2.33). We stress that only the combination (3.11) of these invariants is a genuine invariant of the central-charge extended $N = (1, 1)$ supersymmetry; each of them individually is invariant up to the surface terms. This is analogous to what happens for WZNW or Chern-Simons actions.

Before going further, let us summarize the above discussion. Starting from the scalar $N = (1, 0)$ multiplet u and requiring its fermionic component to be the Goldstone fermion corresponding to spontaneous partial breaking of $N = (1, 1)$ supersymmetry down to $N = (1, 0)$, we uniquely restored the $N = (1, 1)$ supermultiplet to which u should belong. It proved to be a supermultiplet

⁴An equivalent choice of the generators, with Z appearing in both of them, can be achieved by means of the appropriate transformation of the kind (2.7).

of the central-charge extended $N = (1, 1)$ superalgebra (3.3) and it is naturally accommodated by a scalar $N = (1, 1)$ superfield Φ for which the central charge Z generates pure shifts. The invariant action for this system is a combination of two independent invariants, S_1 and S_2 . Thus what we have constructed can be called a superfield model of *linear* realization of the partial spontaneous breaking $N = (1, 1) \rightarrow N = (1, 0)$.

Let us dwell on some peculiar features of this toy model which will show up as well in other $D = 2$ PBGS examples we consider in this paper.

As we saw, the presence of an inhomogeneously transforming fermionic Goldstone component in the scalar supermultiplet of unbroken supersymmetry ($N = (1, 0)$) inevitably implies the appearance of the central charge generator in the algebra of full spontaneously broken supersymmetry ($N = (1, 1)$). The symmetry generated by the central charge is also spontaneously broken, the physical bosonic field of the scalar supermultiplet being the appropriate inhomogeneously transforming Goldstone boson ($N = (1, 0)$ superpartner of Goldstino).

We may reverse the argument by starting from the central-charge extended $N = (1, 1)$ superalgebra (3.3) and introducing the scalar superfield $\Phi(z)$ (3.5) which is shifted by a constant under the action of the central charge generator (eq. (3.6)). So it can be regarded as the Goldstone superfield associated with this spontaneously broken generator. Its supersymmetry transformations are uniquely defined by (3.7), (3.8). One half of these transformations, the $N = (0, 1)$ ones, contain an inhomogeneous shift $i\epsilon^-\theta^+$, which implies that $iD_+\Phi$ is shifted by ϵ^- . Thus $iD_+\Phi$ is the Goldstone fermionic superfield whose presence is tantamount to the spontaneous breakdown of the half of $N = (1, 1)$ supersymmetry. In other words, the spontaneous breaking of the central charge symmetry in (3.3) entails half-breaking of $N = (1, 1)$ supersymmetry. This can be also understood from the following heuristic reasoning: once Z is spontaneously broken, it should not give zero while applied to vacuum; then from the crossing relation in (3.3) follows that \hat{Q}_+ or/and \hat{Q}_- must also possess this property, i.e. it generates a spontaneously broken symmetry.

It is worth mentioning that the transformation properties of Φ as the Goldstone superfield can be rederived on pure geometrical ground proceeding from the coset (nonlinear) realizations method. Let us identify the $N = (1, 1)$ superspace co-ordinates and Φ with the parameters of a particular representative of the supergroup corresponding to the algebra (3.3),

$$G(x, \theta^+, \theta^-) = e^{i(x^+P_+ + x^-P_-)} e^{-\theta^+\hat{Q}_+} e^{-\theta^-\hat{Q}_-} e^{-2i\Phi(x, \theta)Z}. \quad (3.14)$$

Then the left multiplications of this element by $e^{\epsilon^+\hat{Q}_+}$ and $e^{\epsilon^-\hat{Q}_-}$ produce for Φ just the $N = (1, 1)$ transformations with the generators (3.7). The covariant derivatives (3.9), (3.10) can be recovered from the standard Cartan approach applied to (3.14).

Closely related to this discussion is the following phenomenon (featured by the $N = (2, 2)$ case as well). As was already mentioned, the generators (3.7) are defined up to a freedom of changing the Z -frame by the rotation of the type (2.7). In our case this freedom is expressed as

$$\begin{aligned} \hat{Q}, \hat{D} &\Rightarrow \hat{Q}(\alpha), \hat{D}(\alpha) = e^{-\alpha\theta^+\theta^-Z}(\hat{Q}, \hat{D})e^{\alpha\theta^+\theta^-Z}, \\ \Phi &\Rightarrow \Phi(\alpha) = e^{-\alpha\theta^+\theta^-Z}\Phi = \Phi - i\frac{\alpha}{2}\theta^+\theta^-. \end{aligned} \quad (3.15)$$

In particular,

$$\hat{Q}_+(\alpha) = Q_+ + \alpha\theta^-Z, \quad \hat{Q}_-(\alpha) = Q_- + (2 - \alpha)\theta^+Z. \quad (3.16)$$

We see that at $\alpha = 2$ the central charge term is entirely pumped over from \hat{Q}_- to \hat{Q}_+ . As a result, the superfield $\Phi(\alpha = 2)$ undergoes an inhomogeneous shift under the $N = (1, 0)$ supersymmetry, i.e. we are facing the breaking option $N = (1, 1) \rightarrow N = (0, 1)$ on such a superfield. It can be equivalently described in terms of the $N = (0, 1)$ components of $\Phi(\alpha = 2)$. This consideration shows that the notion of the linear off-shell realization of $N = (1, 1)$ supersymmetry breaking patterns is

to some extent conditional: various patterns are related to each other by Z -frame rotations which redefine the transformation law of the basic superfield Φ (any such a redefinition amounts to a constant shift of the auxiliary field in Φ). For an arbitrary parameter α in (3.16) *both* fermionic fields in Φ acquire inhomogeneous pieces in their supersymmetry transformations, so this case corresponds to the totally broken $N = (1, 1)$ supersymmetry off shell. This off-shell equivalence of various breaking options is lifted when passing on shell, or when going to their nonlinear realizations as explained below.

Further discussion will be concentrated on the case $\alpha = 0$ corresponding to the pattern $N = (1, 1) \rightarrow N = (1, 0)$. The model we are considering is described by the free action S_1 and as such contains no dynamics. Adding the invariant S_2 merely changes the algebraic equation for the auxiliary field $B \equiv D_+ \eta^+$ which is then forced to be a constant on shell.⁵ A non-trivial self-interacting model can be nonetheless constructed by re-expressing the $N = (1, 1)$ Goldstone superfield Φ in terms of the $N = (1, 0)$ Goldstone superfield $u(x, \theta^+)$ which accommodates in a minimal way both Goldstone degrees of freedom related to the spontaneously broken \hat{Q}_- and Z generators. This procedure is in a sense similar to passing from the linear sigma model with some internal symmetry to the corresponding nonlinear sigma model.

As the starting point, let us note that a minimal model-independent way to implement the $N = (1, 1)$ supersymmetry spontaneously broken down to $N = (1, 0)$ is to use the universal nonlinear realization approach and to introduce the Goldstone fermion superfield $\psi^-(x^\pm, x^=, \theta^+)$ as the coset parameter associated with the generator \hat{Q}_- , in full analogy with the renowned Volkov-Akulov construction for the case of total spontaneous breaking of supersymmetry [26]. For such a Goldstone fermion superfield one immediately derives the universal nonlinear transformation law

$$\delta\psi^- = \epsilon^- - i\epsilon^- \psi^- \partial_- \psi^- . \quad (3.17)$$

The Goldstone fermion $N = (1, 0)$ superfield in any specific model of the spontaneous breaking $N = (1, 1) \rightarrow N = (1, 0)$ is expected to be related to $\psi^-(x^\pm, x^=, \theta^+)$ by a field redefinition.

For the case of total spontaneous breaking of supersymmetry this universality of the nonlinear-realization Goldstone fermion was proven in [20]. Also, a generic method of constructing linear representations of supersymmetry as nonlinear functions of the single Goldstone fermion and its x -derivatives was worked out there. This approach can be generalized rather straightforwardly to the case of partial breaking, and in [21] this already was done for a few simple PBGS patterns. Here we apply a similar construction and covariantly express the linear superfield representation u, η^+ in terms of the single $N = (1, 0)$ superfield Goldstone fermion and further in terms of the basic scalar Goldstone superfield u .

Following [20], [21], this procedure goes through two steps.

First, we should find the *finite* transformation of the superfields⁶ $\xi^- = iD_+ u$ and η^+ under the spontaneously broken $N = (0, 1)$ supersymmetry with parameter ϵ^- . In our one-parameter case this is straightforward as the infinitesimal transformations (3.2) coincide with the finite ones:

$$\xi'^- \equiv \xi^-(\epsilon^-) = \xi^- + \epsilon^-(1 + D_+ \eta^+) , \quad \eta'^+ \equiv \eta^+(\epsilon^-) = \eta^+ - \epsilon^- \partial_- u . \quad (3.18)$$

The second step is to define new objects $\tilde{\xi}^-, \tilde{\eta}^+$ via the substitution $\epsilon^- \Rightarrow -\psi^-$ in (3.18)

$$\tilde{\xi}^- = \xi^-(-\psi^-) = \xi^- - \psi^-(1 + D_+ \eta^+) , \quad \tilde{\eta}^+ = \eta^+(-\psi^-) = \eta^+ + \psi^- \partial_- u . \quad (3.19)$$

⁵Note that in the presence of such a term the fermionic (first) component of the superfield η^+ acquires on shell an inhomogeneous shift under $N = (1, 0)$ supersymmetry proportional to the constant value of auxiliary field. So in this case $N = (1, 1)$ supersymmetry can get totally broken on shell.

⁶On this stage it is preferable to deal with the superfields on which the central charge Z is vanishing and which hence contain no explicit θ 's in their supersymmetry transformations.

Using the transformation law (3.17) of ψ^- , one can check that the superfields $\tilde{\xi}^-$, $\tilde{\eta}^+$ transform homogeneously and independently of each other under the second supersymmetry

$$\delta\tilde{\xi}^- = -i\epsilon^- \psi^- \partial_- \tilde{\xi}^- , \quad \delta\tilde{\eta}^+ = -i\epsilon^- \psi^- \partial_- \tilde{\eta}^+ . \quad (3.20)$$

Thus it is a covariant constraint to put these superfields equal to zero

$$\begin{cases} \tilde{\xi}^- = 0 \\ \tilde{\eta}^+ = 0 \end{cases} \Rightarrow \begin{cases} \xi^- = \psi^- (1 + D_+ \eta^+) \\ \eta^+ = -\psi^- \partial_- u . \end{cases} \quad (3.21)$$

The system (3.21) can easily be solved for η^+ and ψ^- ,

$$\eta^+ = -\frac{iD_+ u \partial_- u}{1 + D_+^2 \eta^+} \Rightarrow \eta^+(u) = -\frac{2iD_+ u \partial_- u}{1 + \sqrt{1 - 4\partial_- u \partial_+ u}} , \quad (3.22)$$

$$\psi^-(u) = \frac{2iD_+ u}{1 + \sqrt{1 - 4\partial_- u \partial_+ u}} . \quad (3.23)$$

Eq. (3.23) gives the anticipated equivalence relation between the nonlinear-realization Goldstone fermion ψ^- and its linear-realization counterpart $\xi^- = iD_+ u$ ($\psi^- = \xi^- + \dots$). The $N = (1, 0)$ superfield η^+ which completes u to a scalar $N = (1, 1)$ supermultiplet is expressed by eq. (3.22) through u itself. Thus, u remains as the only independent quantity of our theory.

Note that the constraints (3.21) can be reformulated in terms of the $N = (1, 1)$ superfield Φ , but this equivalent form is not too enlightening. We also notice that the expression (3.22) for $\eta^+(u)$ could be derived by imposing proper covariant constraints directly on the covariant derivatives $\hat{D}_\pm \Phi$, without introducing the auxiliary object ψ^- at the intermediate step (this would be in the spirit of the method of refs. [27, 12, 13]). Once again, these constraints look rather involved, and it would be difficult to guess their form. In contrast, the above universal method unambiguously leads to the desired answer.

Finally, we substitute the expression for $\eta^+(u)$ in both our actions (3.12), (3.13) and find that they coincide

$$S_1 = S_2 = \frac{1}{f^2} \int d^3 z^{(+)} \eta^+(u) = \frac{i}{f^2} \int d^2 x D_+ \left(\frac{D_+ u \partial_- u}{1 + \sqrt{1 - 4\partial_- u \partial_+ u}} \right) . \quad (3.24)$$

Thus the action (3.11) with the genuinely invariant Lagrangian density is vanishing for $\Phi(u) = u + i\theta^- \eta^+(u)$. This can be directly seen from the fact that the spinor covariant derivatives of Φ , eqs. (3.10), on the shell of the constraints (3.21) are proportional to the same Grassmann quantity

$$\hat{D}_- \Phi(u) = i(\theta^- + \psi^-) \partial_- u , \quad \hat{D}_+ \Phi(u) = -i(\theta^- + \psi^-)(1 + D_+ \eta^+) , \quad (3.25)$$

and so the Lagrangian density in (3.11) equals to zero for $\Phi(u)$.

In accord with the general concept of the nonlinear-realization method we expect that the Goldstone superfield action (3.24) is universal, in the sense that it describes the low-energy dynamics of any $D = 2$ model where a spontaneous breaking of $N = (1, 1)$ supersymmetry down to $N = (1, 0)$ with a scalar Goldstone multiplet occurs. To reveal its relation to string theory, let us note that for the physical bosonic component $u_0 = u|_{\theta^+ = 0}$ one obtains just the static-gauge form of the Nambu-Goto $D = 3$ string action

$$S_{bos} = \frac{1}{4} \int d^2 x \left(1 - \sqrt{1 - 4\partial_- u_0 \partial_+ u_0} \right) . \quad (3.26)$$

This suggests an interpretation of the Goldstone field u as the transverse string co-ordinate and of the whole superfield action (3.24) as the static-gauge form of the action of the $N = 1, D = 3$ superstring in a flat Minkowski background. Actually, from the $D = 3$ perspective the superalgebra (3.3) is just the $N = 1$ Poincaré superalgebra, with Z being the momentum in the third direction. The component form of the action (3.24) could be recovered from the standard Green-Schwarz action for this superstring, like it has been done in [8] for the PBGS form of the $N = 1, D = 4$ superstring action (we shall reproduce this example in Sect. 5). A novel point of our consideration is that the action (3.24) was constructed directly in $D = 2$ superspace, proceeding only from the purpose to describe the partial breaking $N = (1, 1) \rightarrow N = (1, 0)$ and not assuming any $D = 3$ structure beforehand.

As a final remark, note that we could equally start from the linear realization (3.16) with $\alpha = 2$ which corresponds to the partial breaking option $N = (1, 1) \rightarrow N = (0, 1)$. The relevant nonlinear realization is constructed along the same lines, but with $u(x, \theta^-) = \Phi(\alpha = 2)|_{\theta^+ = 0}$ as the irreducible Goldstone superfield. As a $D = 2$ field theory, it is clearly non-equivalent to the previous one because the Goldstone fermions in both theories have opposite light-cone chiralities (purely bosonic sectors are identical). Nevertheless, their Goldstone superfield actions are related to each other by a kind of mirror symmetry and are gauge-equivalent from the $D = 3$ perspective, corresponding to two different choices of gauge with respect to kappa-symmetry in the same $N = 1, D = 3$ superstring Green-Schwarz action. A nonlinear realization of the total breaking pattern (3.16) with $\alpha \neq 0, 2$ can be straightforwardly constructed by the original methods of [20]. It is a modification of the standard Volkov-Akulov theory for $N = (1, 1), D = 2$ [27], with one extra Goldstone scalar field for the spontaneously broken central charge generator.⁷ Thus various options of the $N = (1, 1)$ breaking, being equivalent modulo Z -frame rotations at the level of the linear realization, yield non-equivalent $D = 2$ theories after passing to the relevant nonlinear realizations.

3.2 $N = (2, 0) \rightarrow N = (1, 0)$

This case is somewhat special. The main peculiarity is that the $(2, 0)$ superalgebra

$$\{Q_+^1, Q_+^1\} = \{Q_+^2, Q_+^2\} = 2P_+ \quad (3.27)$$

admits no $SO(1, 1)$ - scalar central charges. Hence, we do not expect to find a scalar bosonic field with a shift symmetry among the set of our fields. The generators $Q_+^{1,2}$ and covariant derivatives $D_+^{1,2}$ are defined by the standard formulas without central charge terms,

$$Q_+^{1,2} = \frac{\partial}{\partial \theta_{1,2}^+} + i\theta_{1,2}^+ \partial_+, \quad D_+^{1,2} = \frac{\partial}{\partial \theta_{1,2}^+} - i\theta_{1,2}^+ \partial_+. \quad (3.28)$$

We wish to describe the situation where $N = (1, 0)$ supersymmetry generated by Q_+^1 is unbroken while the other $N = (1, 0)$ supersymmetry, with generator Q_+^2 , is spontaneously broken. Thus, like in the preceding Subsection, we are led to introduce a real Goldstone fermionic $N = (1, 0)$ superfield

$$\xi^+(x^\pm, x^-, \theta_1^+) = \lambda^+(x^\pm, x^-) + \theta_1^+ F(x^\pm, x^-) \quad (3.29)$$

($[\xi^+] = -1/2$), which contains a chiral fermion λ^+ and an auxiliary real bosonic field F . As opposed to the previously discussed case we cannot represent ξ^+ as a covariant spinor derivative of some scalar $N = (1, 0)$ superfield since we now have at our disposal only one spinor derivative D_+^1 ($(D_+^1)^2 = -i\partial_+$). Nevertheless, we can proceed in a similar way and, first of all, try to construct a linear realization of the breaking pattern by extending ξ^+ to an $N = (2, 0)$ multiplet. The simplest

⁷Such a modification was considered in [28].

possibility is to add one more fermionic superfield $\eta^+(x^\pm, x^=, \theta^+)$ which can be combined with ξ^+ to an $N = (2, 0)$ supermultiplet. The following transformations

$$\delta\xi^+ = \epsilon_2^+ + \epsilon_2^+ D_+^1 \eta^+ , \quad \delta\eta^+ = -\epsilon_2^+ D_+^1 \xi^+ \quad (3.30)$$

can be checked to form just the Q_+^2 part of the algebra (3.27). These $N = (1, 0)$ superfields are related to a chiral spinor $N = (2, 0)$ superfield Ξ^+ via

$$\Xi^+(x_L^\pm, x^=, \theta_1^+ + i\theta_2^+) = \xi^+ - i\eta^+ + i\theta_2^+ D_+^1 (\xi^+ - i\eta^+) , \quad (3.31)$$

where we expanded Ξ^+ in θ_2^+ by making use of the definition of x_L^\pm in (2.21). Assuming for Ξ^+ the inhomogeneous transformation law under the ϵ_2^+ supersymmetry

$$\delta\Xi^+ = \epsilon_2^+ + (\epsilon_2^+ Q_+^2) \Xi^+ , \quad (3.32)$$

we get for ξ, η just the transformation laws (3.30).

Like in the previous case, one can construct for this supermultiplet two independent off-shell invariants

$$S_1 = \frac{1}{2f^2} \int d^4 z^{(+)} \Xi^+ \bar{\Xi}^+ = -\frac{i}{f^2} \int d^3 z^{(+)} (\xi^+ D_+^1 \xi^+ + \eta^+ D_+^1 \eta^+) , \quad (3.33)$$

$$S_2 = -\frac{1}{2\sqrt{2}f^2} (b + i) \int dx^= d^2 \zeta_L^{(+)} \Xi^+ + \text{c.c.} = \frac{i}{f^2} \int d^3 z^{(+)} (\eta^+ + b\xi^+) , \quad (3.34)$$

where b is an arbitrary real constant. The first invariant describes the free theory of two chiral fermions. Adding the second invariant merely changes the algebraic equations of motion for the auxiliary fields $D_+^1 \eta^+$ and $D_+^1 \xi^+$ allowing them to be non-zero constants. In its presence, the on-shell pattern of supersymmetry breaking differs from the off-shell one we started with. In particular, the entire $N = (2, 0)$ supersymmetry can be broken.

Once again, it is possible to trade the second superfield η^+ for the Goldstone one ξ^+ to obtain the minimal theory in terms of the single superfield ξ^+ with a nonlinearly realized ϵ_2^+ supersymmetry. To this end, we construct the superfields

$$\tilde{\xi}^+ = \xi^+ - \psi^+ (1 + D_+^1 \eta^+) , \quad \tilde{\eta}^+ = \eta^+ + \psi^+ D_+^1 \xi^+ , \quad (3.35)$$

where $\psi^+(x, \theta_1^+)$ is the Goldstone fermion with the universal transformation law

$$\delta\psi^+ = \epsilon_2^+ - i\epsilon_2^+ \psi^+ \partial_\mp \psi^+ \quad (3.36)$$

(cf. eq. (3.17)). These objects transform homogeneously,

$$\delta\tilde{\xi}^+ = -i\epsilon_2^+ \psi^+ \partial_\mp \tilde{\xi}^+ , \quad \delta\tilde{\eta}^+ = -i\epsilon_2^+ \psi^+ \partial_\mp \tilde{\eta}^+ ,$$

and hence can be covariantly equated to zero, leading to the desired expressions of both ψ^+ and η^+ in terms of ξ^+ :

$$\begin{aligned} \tilde{\xi}^+ = \tilde{\eta}^+ = 0 &\Rightarrow \\ \eta^+ = -\frac{2\xi^+ D_+^1 \xi^+}{1 + \sqrt{1 - 4(D_+^1 \xi^+)^2}} , \quad \psi^+ = \frac{2\xi^+}{1 + \sqrt{1 - 4(D_+^1 \xi^+)^2}} . \end{aligned} \quad (3.37)$$

Substituting this expression for η^+ into the invariants (3.33) and (3.34), we find that for $b = 0$ they coincide:

$$S_1 = S_2(b = 0) = -\frac{2i}{f^2} \int d^3 z^{(+)} \left(\frac{\xi^+ D_+^1 \xi^+}{1 + \sqrt{1 - 4(D_+^1 \xi^+)^2}} \right) . \quad (3.38)$$

Adding the term $\sim b \neq 0$ results only in a modification of the equation of motion for the non-propagating field $F = D_+^1 \xi^+$ which becomes a constant $\sim b$ on shell. As a result, the Goldstone component λ^+ starts to transform inhomogeneously under the $(1, 0)$ supersymmetry which was originally unbroken off shell. Clearly, there still exists a combination of the supersymmetry generators under which λ^+ transforms homogeneously, so the effect of partial breaking retains on shell. However, only in the $b = 0$ case the off- and on-shell patterns of this breaking are in one-to-one correspondence.

Despite its nonlinear appearance, the Goldstone superfield action (3.38) in the present case yields trivial dynamics. The component form of (3.38) is as follows

$$S_1 \sim \frac{i\lambda^+ \partial_{\mp} \lambda^+}{(1 + \sqrt{1 - 4F^2})\sqrt{1 - 4F^2}} + \frac{1}{4}(1 - \sqrt{1 - 4F^2}) \quad (3.39)$$

(the component fields were defined in (3.29)). It gives rise to the free equations for the involved fields

$$\partial_{\mp} \lambda^+ = 0, \quad F = 0. \quad (3.40)$$

The absence of interaction can be made manifest by making the invertible field redefinition

$$\mu^+ \equiv \frac{\sqrt{2} \xi^+}{\sqrt{1 + \sqrt{1 - 4(D_+^1 \xi^+)^2}}}, \quad \xi^+ = \mu^+ \sqrt{1 - (D_+^1 \mu^+)^2}, \quad (3.41)$$

after which the action (3.38) is reduced to the free one ,

$$S_1 = -\frac{i}{f^2} \int d^3 z^{(+)} \mu^+ D_+^1 \mu^+. \quad (3.42)$$

Thus, the $N = (2, 0) \rightarrow N = (1, 0)$ PBGS in $D = 2$ corresponds to a degenerate system of one free left-chiral fermion $\lambda^+(x)$ ($\lambda^+ = \lambda^+(x^-)$ on shell).

It is worth mentioning that, in accord with the well-known $D = 2$ equivalence between the abelian gauge field strength and an auxiliary field, the fermionic superfield ξ^+ can also be treated as a covariant field strength for the $N = (1, 0)$ fermionic and bosonic “gauge potentials” A_+ and A_- [2]:

$$\xi^+ \Rightarrow \xi_v^+ = \partial_- A_+ - D_+^1 A_- , \quad \delta A_+ = D_+^1 \Lambda , \quad \delta A_- = \partial_- \Lambda . \quad (3.43)$$

In the Wess-Zumino gauge ξ_v^+ has the same component content as in (3.29), with the auxiliary F substituted by the field strength of the $D = 2$ vector gauge field (v_+, v_-)

$$F \Rightarrow F_v = \partial_+ v_- - \partial_- v_+ . \quad (3.44)$$

The supersymmetry transformation properties of ξ_v^+ do not change compared to those of ξ^+ , and the invariant action is still given by (3.38) and (3.39). The action of the fermionic field becomes free after a proper field rescaling, while the F_v part is recognized as the $D = 2$ Born-Infeld action

$$S(F_v) \sim \int d^2 x \left(1 - \sqrt{1 - 4F_v^2} \right) . \quad (3.45)$$

Thus the model of partial breaking $N = (2, 0) \rightarrow N = (1, 0)$ with the $N = (1, 0)$ vector Goldstone multiplet amounts to a sort of “space-filling” $N = (2, 0)$ D1-brane, which has just this multiplet as the physical world-sheet one. Since the gauge field is non-dynamical in $D = 2$ at the classical level irrespective of its action, we are still left with a free theory of one chiral fermion in this case.

Note that at the quantum level or for non-vanishing first Chern number the abelian gauge field is non-trivial [5]. In this connection, it is worthwhile to mention that the term $\sim b$ in (3.34) yields just

the well-known topological invariant of the $D = 2$ abelian gauge field upon the substitution (3.44). The chiral $N = (2, 0)$ superfield Ξ^+ (3.31) with η^+ covariantly expressed through ξ_v^+ by (3.37) can be identified with the superfield strength of the $N = (2, 0)$ gauge vector multiplet [2, 3, 29]

$$\Xi_v^+(x, \theta^+) = \partial_- \bar{D}_+ V - \bar{D}_+ A_- , \quad (3.46)$$

where the real $N = (2, 0)$ superfield V and antichiral superfield A_- , $D_+ A_- = 0$, are the corresponding gauge potentials

$$\delta V = a + \bar{a} , \quad \delta A_- = \partial_- a , \quad D_+ a = 0 .$$

The coefficient in front of the $N = (2, 0)$ superspace integral in (3.34) is recognized as the familiar complex coupling constant [5, 6] combining the coefficient of the Fayet-Iliopoulos term (η_v^+ in (3.34)) and the well-known θ -angle:

$$\frac{1}{f^2}(i + b) \equiv \tau = ir + \frac{\theta}{2\pi} .$$

Thus the parameter b has the physical meaning of a θ -angle.

Finally, let us notice that all models of this and the following Sections submit to the argument used in [8, 9] to evade the partial breaking no-go theorem of [30]. Namely, the Noether energy-momentum tensors corresponding to the $D = 2$ translation generators in the algebras of broken and unbroken supersymmetries do not coincide, but differ by some constants. These constant “central charges” should not be confused with the active central charges which act as shifts of the Goldstone superfields and the presence of which in many cases turns out to be crucial for triggering the partial spontaneous breakdown of supersymmetry. Also, to avoid a possible misunderstanding, we point out that all superalgebras used throughout this paper should be regarded as algebras of infinitesimal field transformations rather than as algebras of the charge and supercharge generators computed by the Noether procedure from some invariant action. For spontaneously broken symmetries and supersymmetries the latter objects are often ill-defined, while the algebras of the corresponding currents and supercurrents are always meaningful.

4 Partial breaking $N=(2,2)$ to $N=(1,1)$

We start the study of the partial breaking pattern $N = (2, 2) \rightarrow N = (1, 1)$ by constructing its linear realization. Without loss of generality, we let Q_\pm^1 generate the unbroken $N = (1, 1)$ supersymmetry.

A simple analysis shows that the scalar real $N = (1, 1)$ superfield $\pi(x, \theta_1^+, \theta_1^-)$ has the components content just appropriate for the relevant Goldstone supermultiplet, including two real fermionic fields $\kappa^\pm(x)$ capable to be the Goldstone fermions associated with the spontaneously broken Q_\pm^2 supersymmetry:

$$\pi = p(x) + i\theta_1^+ \kappa^-(x) + i\theta_1^- \kappa^+(x) + i\theta_1^+ \theta_1^- F(x) . \quad (4.1)$$

Let us show that this multiplet can be naturally embedded into a chiral scalar $N = (2, 2)$ superfield $\phi(x_L^\pm, x_L^\mp, \theta^+, \theta^-)$ with the appropriate transformation law including central-charge terms. It is similar to the transformation laws (3.6), (3.8) of the Goldstone superfield (3.5) in the linear realization of the partial breaking $N = (1, 1) \rightarrow N = (1, 0)$ considered in Subsect. 3.1.

We shall need the structure of the $N = (2, 2)$ supersymmetry generators with central charges in the realization on chiral superfields. Recalling the discussion after eq. (2.25), we are led to choose the frame where the spinor derivatives \bar{D}_\pm contain no central charge terms and where the standard chirality constraints (2.24) are valid. Bearing also in mind that in $N = (2, 2)$ supersymmetry with central charges the chiral superfields can be defined only under the restriction $Z_3 - iZ_4 = 0$ in (2.5)

- (2.6) and (2.9) - (2.18) (recall eq. (2.26)), the explicit form of the $N = (2, 2)$ generators in this frame is as follows

$$\begin{aligned}
\hat{Q}_+ &= \frac{\partial}{\partial \theta^+} + i\bar{\theta}^+ \partial_{\mp} - \theta^-(Z_3 + iZ_4) \equiv Q_+ - \theta^-(Z_3 + iZ_4) , \\
\hat{\bar{Q}}_+ &= \frac{\partial}{\partial \bar{\theta}^+} + i\theta^+ \partial_{\mp} + 2\theta^-(Z_1 - iZ_2) \equiv \bar{Q}_+ + 2\theta^-(Z_1 - iZ_2) , \\
\hat{Q}_- &= \frac{\partial}{\partial \theta^-} + i\bar{\theta}^- \partial_{=} - \theta^+(Z_3 + iZ_4) \equiv Q_- - \theta^+(Z_3 + iZ_4) , \\
\hat{\bar{Q}}_- &= \frac{\partial}{\partial \bar{\theta}^-} + i\theta^- \partial_{=} + 2\theta^+(Z_1 + iZ_2) \equiv \bar{Q}_- + 2\theta^+(Z_1 + iZ_2) ,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\delta\phi &= (\epsilon\hat{Q})\phi = (\epsilon Q)\phi + 2\bar{\epsilon}^+\theta^-(Z_1 - iZ_2)\phi + 2\bar{\epsilon}^-\theta^+(Z_1 + iZ_2)\phi \\
&\quad - (\epsilon^+\theta^- + \epsilon^-\theta^+)(Z_3 + iZ_4)\phi .
\end{aligned} \tag{4.3}$$

Like in the $N = (1, 1) \rightarrow N = (1, 0)$ case, in order to trigger the spontaneous breaking of $N = (2, 2)$ to $N = (1, 1)$ the superfield ϕ is expected to undergo pure shifts under the action of the central charge generators Z_1, Z_2 . Since ϕ is complex, it can be shifted by a complex parameter, i.e. in principle both Z_1 and Z_2 can be non-vanishing on ϕ . However, we wish to have the $N = (1, 1)$ generators

$$\hat{Q}_{\pm}^1 \sim \hat{Q}_{\pm} + \hat{\bar{Q}}_{\pm} = Q_{\pm} + \bar{Q}_{\pm} - \theta^{\mp}(Z_3 + iZ_4) + 2\theta^{\mp}(Z_1 \mp iZ_2)$$

unbroken, i.e. having no central charge terms. We still have a freedom of changing the Z -frame according to the rule (2.7),

$$(D, \hat{Q}) \Rightarrow e^{-A}(D, \hat{Q})e^A, \quad \phi \Rightarrow \hat{\phi} = e^{-A}\phi \tag{4.4}$$

with $A \sim \theta^+\theta^- \times [\text{central charges}]$. Such a transformation commutes with \bar{D}_{\pm} and so does not affect the chirality condition (2.24) (it amounts to a redefinition of the auxiliary field in ϕ). At the same time it gives rise to the appearance of the central charge terms in the generators \hat{Q}_{\pm} , leaving intact $\hat{\bar{Q}}_{\pm}$. A simple inspection shows that the necessary and sufficient conditions for the central charge terms to drop out from \hat{Q}_{\pm}^1 , modulo the freedom (4.4), is the following one,

$$[2Z_1 - (Z_3 + iZ_4)]\phi = 0 \tag{4.5}$$

(together with (2.26), it implies $Z_1 = Z_3$). Then, the rotation (4.4) with

$$A = 2i\theta^+\theta^- Z_2 \tag{4.6}$$

eliminates the central charge terms from \hat{Q}_{\pm}^1 .

There exist several possibilities to realize the breaking pattern $N = (2, 2) \rightarrow N = (1, 1)$ on $\hat{\phi}$. In particular, one could keep two independent central charges producing a complex shift of $\hat{\phi}$, with two Goldstone-type $N = (1, 1)$ supermultiplets. We are interested in the minimal possibility with a single Goldstone multiplet comprised by the $N = (1, 1)$ superfield (4.1). The choice of central charges giving rise to this option can be shown to be as follows,

$$Z_1\phi = Z_3\phi = Z_4\phi = 0, \quad Z_2\phi = -\frac{1}{2\sqrt{2}}, \tag{4.7}$$

$$\delta_z\phi = -2\sqrt{2}ia[Z_2, \phi] = ia. \tag{4.8}$$

Then, for the rotated chiral superfield

$$\hat{\phi} = e^{-A}\phi = \phi - 2i\theta^+\theta^- Z_2\phi = \phi + \frac{i}{\sqrt{2}}\theta^+\theta^- \tag{4.9}$$

we get the following transformation law

$$\delta\hat{\phi} = (\epsilon\hat{Q})\hat{\phi} = \epsilon_2^+\theta^- - \epsilon_2^-\theta^+ + (\epsilon Q)\hat{\phi} . \quad (4.10)$$

Here the generators \hat{Q} are related to the original ones (4.2) by the rotation (4.4) with A (4.6) (using the same notation for these two sets of generators will hopefully not lead to a confusion).

A few remarks are in order at this stage.

Firstly, let us point out that in the presence of all central charge generators in (4.2) acting as shifts of ϕ , only one real combination of the $N = (2, 2)$ supercharges can be arranged to include no central charge terms by performing an appropriate Z -frame rotation. As a result, in this case there becomes possible the 1/4 partial breaking of $N = (2, 2)$ supersymmetry down to its $N = (1, 0)$ (or $N = (0, 1)$) subgroup, with three out of four real fermionic component fields of ϕ having inhomogeneous transformation laws and so being Goldstone fermions. This option will be considered in Section 6.

Secondly, we could choose

$$Z_2\phi = 0 , \quad Z_1\phi \neq 0 , \quad (Z_3 + iZ_4)\phi = 0$$

instead of (4.7). In this case it is possible to remove central charge terms from the $N = (1, 1)$ generators $\hat{Q}_+^1, \hat{Q}_-^2 \sim i(\hat{Q}_- - \hat{Q}_+)$ (or \hat{Q}_-^1, \hat{Q}_+^2). This option amounts to an equivalent pattern of the partial breaking of $N = (2, 2)$ supersymmetry to $N = (1, 1)$.

As a last remark we note that it is possible to construct the real $N = (2, 2)$ superfield basically as a sum of $\hat{\phi}$ and $\hat{\bar{\phi}}$. It is invariant under Z_2 shifts and, as a consequence, has the standard homogeneous transformation properties under the full $N = (2, 2)$ supersymmetry. To construct it, one should make the similarity rotation back to the central-basis frame of $N = (2, 2)$ superspace where the generators \hat{Q}_\pm and $\hat{\bar{Q}}_\pm$ become conjugated to each other in the ordinary sense. The chiral $N = (2, 2)$ superfield is “covariantly chiral” in such a frame. The precise relation between two frames is given by

$$\phi_{cov} = e^B\hat{\phi} = \hat{\phi} + \frac{i}{2\sqrt{2}}(\theta^+\bar{\theta}^- + \bar{\theta}^+\theta^- - 2\theta^+\theta^-) , \quad (4.11)$$

with

$$B = -i(\theta^+\bar{\theta}^- + \bar{\theta}^+\theta^- - 2\theta^+\theta^-) Z_2 .$$

It is straightforward to check that the real superfield

$$\hat{\Phi} \equiv 2Re\phi_{cov} = \hat{\phi} + \hat{\bar{\phi}} + i\sqrt{2}\theta_2^+\theta_2^- \quad (4.12)$$

possesses zero central charge and transforms in the conventional way under $N = (2, 2)$ supersymmetry

$$\delta\hat{\Phi} = (\epsilon Q)\hat{\Phi} . \quad (4.13)$$

Here Q_\pm, \bar{Q}_\pm contain no central charge terms.

In our further exposition we shall closely follow the lines we pursued for the toy examples in the previous Section.

We need to know how $\hat{\phi}$ is expressed through $N = (1, 1)$ superfields appearing in its (θ_2^+, θ_2^-) expansion. The chirality conditions (2.24) rewritten in the real basis

$$(D_+^1 + iD_+^2)\phi = (D_-^1 + iD_-^2)\phi = 0 \quad (4.14)$$

can be solved in terms of the complex $N = (1, 1)$ superfield $\chi = \frac{1}{\sqrt{2}}(\sigma + i\pi)$ as follows

$$\phi(x_L, \theta^\pm) = \frac{1}{\sqrt{2}} \left[1 + i\theta_2^+ D_+^1 + i\theta_2^- D_-^1 + \theta_2^+ \theta_2^- D_+^1 D_-^1 \right] \left[\sigma(x, \theta_1^\pm) + i\pi(x, \theta_1^\pm) \right] . \quad (4.15)$$

The infinitesimal $N = (2, 2)$ superfield transformation (4.10) implies, via the relation (4.15), the following transformation laws for the real $N = (1, 1)$ superfields σ, π :

$$\begin{aligned}\delta\pi &= i(\epsilon_2^- \theta_1^+ - \epsilon_2^+ \theta_1^-) + (\epsilon_2^+ D_+^1 + \epsilon_2^- D_-^1)\sigma , \\ \delta\sigma &= -(\epsilon_2^+ D_+^1 + \epsilon_2^- D_-^1)\pi .\end{aligned}\quad (4.16)$$

In the closure of these transformations one finds a pure shift of π generated by Z_2 , so π can be interpreted as the $N = (1, 1)$ Goldstone superfield for the spontaneously broken central charge transformations. Its spinor derivatives,

$$\xi^- \equiv iD_+^1 \pi , \quad \xi^+ \equiv -iD_-^1 \pi , \quad (4.17)$$

also transform inhomogeneously

$$\delta\xi^- = \epsilon_2^- (1 - iD_+^1 D_-^1 \sigma) - \epsilon_2^+ \partial_+ \sigma , \quad \delta\xi^+ = \epsilon_2^+ (1 - iD_+^1 D_-^1 \sigma) + \epsilon_2^- \partial_- \sigma , \quad (4.18)$$

and so they are the relevant linear-realization Goldstone fermions for the spontaneously broken half of $N = (2, 2)$ supersymmetry.

Like in the previous examples, in this linear realization all invariants one can construct correspond to a free theory. In the $N = (2, 2)$ and $N = (1, 1)$ superspaces they are represented by the following expressions

$$S_1 = \frac{1}{2f^2} \int d^6 z \phi \bar{\phi} = -\frac{i}{f^2} \int d^4 z (D_+^1 \sigma D_-^1 \sigma + D_+^1 \pi D_-^1 \pi) , \quad (4.19)$$

$$S_2 = -\frac{(b+i)}{\sqrt{2}f^2} \int d^4 \zeta_L \phi + \text{h.c.} = \frac{1}{f^2} \int d^4 z (\sigma + b\pi) . \quad (4.20)$$

The next step is to construct the nonlinear realization of the spontaneous breaking $N = (2, 2) \rightarrow N = (1, 1)$ in terms of the sole scalar Goldstone multiplet $\pi(x, \theta_1^\pm)$. As before, we should firstly construct the $N = (1, 1)$ superfields with a homogeneous transformation law under the spontaneously broken supersymmetry. This can be done according to the universal prescription, by substituting the universal nonlinear realization Goldstone fermion $N = (1, 1)$ superfields ψ^\pm for the supergroup parameters in the finite transformation laws. These Goldstone fermions transform according to

$$\delta\psi^+ = \epsilon_2^+ - i(\epsilon_2^+ \psi^+ \partial_+ + \epsilon_2^- \psi^- \partial_-) \psi^+ , \quad \delta\psi^- = \epsilon_2^- - i(\epsilon_2^+ \psi^+ \partial_+ + \epsilon_2^- \psi^- \partial_-) \psi^- . \quad (4.21)$$

Denoting the finite transforms of ξ^\pm and σ by $\tilde{\xi}^\pm(\epsilon_2^+, \epsilon_2^-)$, $\tilde{\sigma}(\epsilon_2^+, \epsilon_2^-)$ (these quantities can be straightforwardly constructed by the infinitesimal laws (4.16), (4.18)), the constraints read

$$\tilde{\xi}^\pm(-\psi^+, -\psi^-) = 0 , \quad \tilde{\sigma}(-\psi^+, -\psi^-) = 0 . \quad (4.22)$$

Once again, these are covariant with respect to both unbroken and broken supersymmetries since the $N = (1, 1)$ superfields on their l.h.s. transform homogeneously under the ϵ_2 transformations:

$$\delta\tilde{\xi}^\pm = -i(\epsilon_2^+ \psi^+ \partial_+ + \epsilon_2^- \psi^- \partial_-) \tilde{\xi}^\pm , \quad \tilde{\sigma} = -i(\epsilon_2^+ \psi^+ \partial_+ + \epsilon_2^- \psi^- \partial_-) \tilde{\sigma} .$$

Explicitly, eqs. (4.22) are as follows

$$\begin{aligned}D_+^1 \pi + i\psi^- (1 - iD_+^1 D_-^1 \sigma) - i\psi^+ \partial_+ \sigma - i\psi^+ \psi^- \partial_+ D_-^1 \pi &= 0 , \\ D_-^1 \pi - i\psi^+ (1 - iD_+^1 D_-^1 \sigma) - i\psi^- \partial_- \sigma + i\psi^+ \psi^- \partial_- D_+^1 \pi &= 0 ,\end{aligned}\quad (4.23)$$

$$\sigma + \psi^+ D_+^1 \pi + \psi^- D_-^1 \pi + i\psi^+ \psi^- (1 - iD_+^1 D_-^1 \sigma) = 0 . \quad (4.24)$$

The first two equations set the equivalence relation between the linear- and nonlinear-realizations Goldstone fermions, while the third one (most essential) serves to nonlinearly express the superfield σ through the only independent remaining superfield, the scalar Goldstone superfield π . Note that the linear-realization Goldstone fermions $\sim D_{\pm}^1 \pi$ obey the obvious integrability condition $D_+^1(D_-^1 \pi) = -D_-^1(D_+^1 \pi)$, so their counterparts ψ^{\pm} , in virtue of the relations (4.23), also obey some covariant condition reducing their component content to that of a scalar $N = (1, 1)$ multiplet. We should not care about this since ψ^{\pm} are auxiliary objects which appear only at the intermediate step.

The expression for $\sigma(v)$ can be found rather easily, observing that the relations (4.23) imply

$$\begin{aligned} i\psi^+ \psi^- (1 - iD_+^1 D_-^1 \sigma) &= -\psi^+ D_+^1 \pi = -\psi^- D_-^1 \pi \quad \Rightarrow \\ \sigma(v) &= -\psi^+ D_+^1 \pi = -\psi^- D_-^1 \pi = -\frac{1}{2} \left(\psi^+ D_+^1 \pi + \psi^- D_-^1 \pi \right) . \end{aligned} \quad (4.25)$$

Then simple manipulations using the nilpotency of the fermionic superfields give

$$\sigma(\pi) = -\frac{i D_+^1 \pi D_-^1 \pi}{1 - i D_+^1 D_-^1 \sigma} . \quad (4.26)$$

From this relation it is easy to find the “effective” part of $D_+^1 D_-^1 \sigma$ containing no nilpotent quantities $D_{\pm}^1 \pi$ and to obtain the final expression

$$\sigma(\pi) = -\frac{2i D_+^1 \pi D_-^1 \pi}{1 + \sqrt{1 - 4W}} , \quad W = D_+^1 D_-^1 (D_+^1 \pi D_-^1 \pi) . \quad (4.27)$$

Now we can treat the transformation

$$\delta \pi = i(\epsilon_2^- \theta_1^+ - \epsilon_2^+ \theta_1^-) + (\epsilon_2^+ D_+^1 + \epsilon_2^- D_-^1) \sigma(\pi) \quad (4.28)$$

as a specific form of the nonlinear realization and construct the Goldstone $N = (2, 2)$ superfield $\hat{\phi}$ in this parametrization,

$$\hat{\phi}(\pi) = \frac{1}{\sqrt{2}} \left[1 + i\theta_2^+ D_+^1 + i\theta_2^- D_-^1 + \theta_2^+ \theta_2^- D_+^1 D_-^1 \right] (\sigma(\pi) + i\pi) . \quad (4.29)$$

It is worth noting that the same nonlinear realization of the partial breaking $N = (2, 2) \rightarrow N = (1, 1)$ can be recovered by imposing, in the spirit of refs. [12, 13], the nilpotency condition on the homogeneously transforming real $N = (2, 2)$ superfield $\hat{\Phi}$ defined in (4.12),

$$\hat{\Phi}^2 = 0 . \quad (4.30)$$

In terms of $N = (1, 1)$ superfields π and σ this $N = (2, 2)$ superfield is expressed as

$$\hat{\Phi} \sim \sigma - \theta_2^+ D_+^1 \pi - \theta_2^- D_-^1 \pi + i\theta_2^+ \theta_2^- (1 - iD_+^1 D_-^1 \sigma) \quad (4.31)$$

and constraint (4.30) implies for them that

$$\sigma^2 = 0 , \quad \sigma D_{\pm}^1 \pi = 0 , \quad \sigma = -\frac{i D_+^1 \pi D_-^1 \pi}{1 - i D_+^1 D_-^1 \sigma} .$$

Let us turn to constructing invariant actions for the Goldstone superfield ϕ . It is straightforward to check that

$$D_+^1 \sigma(\pi) D_-^1 \sigma(\pi) + D_+^1 \pi D_+^1 \pi = i\sigma(\pi) . \quad (4.32)$$

Hence, similarly to the previous cases, the invariants S_1 and S_2 in (4.19) and (4.20) basically coincide with each other,

$$S_1 = S_2(b=0) = \frac{1}{f^2} \int d^4z \sigma(\pi) = -\frac{2i}{f^2} \int d^4z \left(\frac{D_+^1 \pi D_-^1 \pi}{1 + \sqrt{1 - 4W}} \right) . \quad (4.33)$$

The most general Goldstone superfield action includes a non-zero parameter b ,

$$S = S(b) = \frac{1}{f^2} \int d^4z (\sigma(\pi) + b \pi) . \quad (4.34)$$

As opposed to the toy example of breaking $N = (2, 0) \rightarrow N = (1, 0)$ (Sect. 3.2), this action necessarily contain a nonpolynomial self-interaction, so adding the term $\sim b$ can have an impact on the associated dynamics. However, in the presence of such a term the original unbroken $N = (1, 1)$ supersymmetry gets broken on shell: the auxiliary field $F = iD_+^1 D_-^1 \pi|$ acquires a non-zero constant part $\sim b$, thus giving rise to inhomogeneous pieces in the ϵ_1 transformations of the Goldstone fermions κ^\pm . Though on shell there still exist two unbroken linear combinations of the $N = (2, 2)$ generators, the off- and on-shell patterns of the partial breaking prove to be different. If we wish these options to coincide, the term $\sim b$ can be ignored.

To establish links with string theory, let us examine the bosonic sector of the Goldstone superfield action (4.33):

$$S_1^{bos} = \frac{1}{2f^2} \int d^2x \left[1 - \sqrt{1 - 4(\partial_+ p \partial_- p + F^2)} \right] . \quad (4.35)$$

The equation of motion for the auxiliary field F implies

$$F = 0 . \quad (4.36)$$

After substituting this into (4.35), the latter is reduced to the static-gauge Nambu-Goto action for a string in $D = 3$ Minkowski space, with $p(x)$ being the corresponding transverse coordinate. From the $D = 3$ standpoint, the $N = (2, 2)$, $D = 2$ superalgebra (2.17) with $Z_1 = Z_3 = Z_4 = 0$ is the $N = 2$ Poincaré superalgebra, with Z_2 completing the pair (P_-, P_+) to the full $D = 3$ energy-momentum vector. Indeed, combining the real generators Q_\pm^1, Q_\pm^2 into two real $SL(2, R)$ spinors,

$$(Q_+^1, Q_-^2) \equiv Q_\alpha , \quad (-Q_+^2, Q_-^1) \equiv S_\alpha ,$$

we can rewrite (2.17) in this particular case as

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{S_\alpha, S_\beta\} = 2P_{\alpha\beta} , \quad P_{11} = P_+, \quad P_{22} = P_- , \quad P_{12} = P_{21} = Z_2 , \\ \{Q_\alpha, S_\beta\} &= 0 . \end{aligned} \quad (4.37)$$

Thus (4.33) can be interpreted as a static-gauge form of the action for an $N = 2$, $D = 3$ superstring in the formulation with manifest world-sheet supersymmetry. It could be recovered by the world-volume dimensional reduction from the PBGS $D = 3$ action describing the $N = 1$, $D = 4$ supermembrane [15]⁸.

Similarly to the $N = (2, 0) \rightarrow N = (1, 0)$ breaking case (Subsect. 3.2), the scalar Goldstone superfield π can be replaced by the covariant superfield strength π_v of the $N = (1, 1)$ vector multiplet [3]

$$\pi \Rightarrow \pi_v = D_+^1 V_- + D_-^1 V_+ . \quad (4.38)$$

⁸The general case (4.5) amounts to an extension of (4.37) by tensorial central charges which is a reduction of the tensorial-charge extension of $N = 1$, $D = 4$ Poincaré superalgebra [22]-[24].

Here, the real fermionic superfields V_{\pm} are the $N = (1, 1)$ gauge potentials,

$$\delta V_{\pm} = D_{\pm}^1 \Lambda .$$

Like in the case considered in Sect. 3.2, π_v in the Wess-Zumino gauge differs from π (3.29) merely by the replacement

$$F \Rightarrow F_v = \partial_{\neq} v_{=} - \partial_{=} v_{\neq} , \quad (4.39)$$

where $(v_{=}(x), v_{\neq}(x))$ is a $D = 2$ abelian gauge field. All the above transformation properties and formulas remain valid. The bosonic action (4.35), with F_v substituted for the auxiliary field F is the $D = 2$ Dirac-Born-Infeld (DBI) action:

$$S_1^{bos}(v) = \frac{1}{2\tilde{f}^2} \int d^2x \left[1 - \sqrt{1 - 4(\partial_{\neq} p \partial_{=} p + F_v^2)} \right] . \quad (4.40)$$

So in this case the superfield action (4.33) can be interpreted as a manifestly world-sheet supersymmetric form of the action of a D1-brane in $D = 3$. It can equally be reproduced by a world-volume dimensional reduction of the space-filling D2-brane [15].

Note that the nonlinear realization of partial breaking $N = (2, 2) \rightarrow N = (1, 1)$ with the Goldstone $N = (1, 1)$ gauge multiplet can also be derived from the same $N = (2, 2)$ superfield transformation law (4.10) as the starting point, but with $\hat{\phi}$ replaced by the chiral superfield strength of $N = (2, 2)$ vector multiplet,

$$\hat{\phi} \rightarrow \hat{\phi}_v = \bar{D}_+ \bar{D}_- \mathcal{V} . \quad (4.41)$$

Here, \mathcal{V} is a real gauge prepotential [3, 4],

$$\delta \mathcal{V} = \lambda + \bar{\lambda} , \quad D_+ \lambda = \bar{D}_- \lambda = 0 . \quad (4.42)$$

The transformation (4.10) for $\hat{\phi}_v$ is induced by the following modified transformation of \mathcal{V} ,

$$\delta \mathcal{V} = - \left[(\epsilon_2^+ \theta^- - \epsilon_2^- \theta^+) \bar{\theta}^- \bar{\theta}^+ + \text{h.c.} \right] + (\epsilon Q) \mathcal{V} .$$

While written through $\phi_v(\pi_v)$, the invariant (4.20) is recognized as a generalized Fayet-Iliopoulos term, with the parameter b being the θ -angle (cf. the discussion in Sect. 3.2).

As the classical Born-Infeld dynamics is trivial in $D = 2$, one can expect the above D1-model to be related to the previous $N = 2, D = 3$ superstring model with the world-sheet scalar multiplet $\pi(x, \theta^{\pm})$. Let us elaborate on this relationship at the level of bosonic actions. The equations for the gauge field in (4.40) yield

$$F_v = \gamma \sqrt{1 - 4(\partial_{\neq} p \partial_{=} p + F_v^2)} , \quad (4.43)$$

where γ is an arbitrary integration constant. Solving this for F_v and substituting the result back into (4.40) brings the latter to the form

$$S_1^{bos}(v) = \frac{1}{2\tilde{f}^2} \int d^2x \left[1 - \sqrt{1 - 4\partial_{\neq} p \partial_{=} p} + \frac{4\gamma^2}{1 + \sqrt{1 - 4\partial_{\neq} p \partial_{=} p}} \right] . \quad (4.44)$$

This coincides with the Nambu-Goto action up to a shift by a positive constant. This shift is the net effect of the presence of the $D = 2$ gauge fields in the initial DBI action (4.40) (at the classical level). Precisely the same bosonic action can be recovered after elimination of the auxiliary field F in the modified $D = 3$ superstring PBGS action (4.34) with $b = -\gamma$. Thus we conclude that the $D = 3$ D1-brane world-sheet effective action $S_1(v)$ is classically equivalent on shell to the modified

$N = 2$, $D = 3$ superstring PBGS action (4.34). Adding the topological θ -term $\sim \frac{\theta}{2\pi} F_v$ to (4.40) does not affect this conclusion.

As a last remark we note that the ϵ_2 transformations of the gauge potentials, which produce for π_v (4.38) the nonlinear-realization transformation law (4.28), are as follows

$$\delta V_+ = i\epsilon_2^- \theta_1^+ \theta_1^- - \epsilon_2^- \sigma(\pi_v) , \quad \delta V_- = -i\epsilon_2^+ \theta_1^+ \theta_1^- - \epsilon_2^+ \sigma(\pi_v) . \quad (4.45)$$

It is easy to check that the algebra of $N = (2, 2)$ supersymmetry is essentially modified on the gauge potentials V_\pm as compared to the gauge-invariant field strength π_v . An analogous modification of the algebra of spontaneously broken supersymmetry on gauge potentials was earlier found for the D3-brane [11, 31] and D2-brane [32] in the PBGS formulations.

5 Partial breaking $N=(2,2)$ to $N=(2,0)$

The partial spontaneous breaking $N = (2, 2) \rightarrow N = (2, 0)$ has been considered earlier in ref. [8]. We shall discuss this case as an illustration of the general methods.

Our starting point will be again the chiral $N = (2, 2)$ superfield ϕ with the generic transformation properties (4.3), (4.2). This time ϕ is required to obey a sort of the holomorphy condition with respect to the central charges

$$(Z_1 - iZ_2)\phi = 0 . \quad (5.1)$$

Then, by means of the appropriate frame rotation, the generators (4.2) can be brought into the form

$$\begin{aligned} \hat{Q}_+ &= Q_+ , \quad \hat{\bar{Q}}_+ = \bar{Q}_+ , \\ \hat{Q}_- &= Q_- - 2\theta^+(Z_3 + iZ_4) , \quad \hat{\bar{Q}}_- = \bar{Q}_- + 2\theta^+(Z_1 + iZ_2) . \end{aligned} \quad (5.2)$$

The $N = (2, 0)$ generators \hat{Q}_+ , $\hat{\bar{Q}}_+$ contain no central charge terms and hence correspond to unbroken supersymmetry. For the $N = (0, 2)$ part of $N = (2, 2)$ supersymmetry to be fully broken, we have two obvious alternatives

$$(a) (Z_1 + iZ_2)\phi = 0 , \quad (Z_3 + iZ_4)\phi \neq 0; \quad (b) (Z_1 + iZ_2)\phi \neq 0 , \quad (Z_3 + iZ_4)\phi = 0 . \quad (5.3)$$

Note that in the case of simultaneous presence of both $Z_1 + iZ_2$ and $Z_3 + iZ_4$ acting as shifts of ϕ with the relative coefficient being a pure phase, one can always find a real combination of the generators \hat{Q}_- , $\hat{\bar{Q}}_-$ containing no central charges. In this case we are facing the 3/4 partial breaking option, with only one real $N = (0, 1)$ supersymmetry being broken. This interesting option deserves a special analysis which is however beyond the scope of the present paper.⁹

If the relative coefficient is not a pure phase, $N = (0, 2)$ supersymmetry is fully broken, i.e. we are facing the 1/2 breaking of $N = (2, 2)$. We shall not consider this most general mixed case of the $N = (2, 2) \rightarrow N = (2, 0)$ partial breaking, but limit our study to the above two extremal cases for simplicity.

These two versions give rise to two different ϵ^- transformation laws for ϕ

$$(a) \delta\phi = \epsilon^- \theta^+ + (\epsilon^- Q_- + \bar{\epsilon}^- \bar{Q}_-) \phi , \quad (Z_3 + iZ_4)\phi = -\frac{1}{2} \quad (5.4)$$

$$(b) \delta\phi = \bar{\epsilon}^- \theta^+ + (\epsilon^- Q_- + \bar{\epsilon}^- \bar{Q}_-) \phi , \quad (Z_1 + iZ_2)\phi = \frac{1}{2} . \quad (5.5)$$

⁹Note that this possibility arises only if two sorts of complex central charge generators are simultaneously present in the $N = (2, 2)$ superalgebra (2.5), (2.17). This is in agreement with the general conclusions of refs. [22]-[24] that such an option in the case of $N = 1$, $D = 4$ supersymmetry is possible only in the presence of tensorial central charges (see the footnote after eq. (2.18)).

In both of them two ϵ^- supersymmetries are broken with $\xi^- \equiv -D_+\phi$ as the corresponding Goldstone fermion

$$(a) \delta\xi^- = \epsilon^- + (\epsilon^- Q_- + \bar{\epsilon}^- \bar{Q}_-) \xi^- , \quad (b) \delta\xi^- = \bar{\epsilon}^- + (\epsilon^- Q_- + \bar{\epsilon}^- \bar{Q}_-) \xi^- . \quad (5.6)$$

The central charges shift ϕ by a complex parameter. Like in the previously considered examples, the specific realization of the central charges on ϕ in (5.4), (5.5) was chosen for convenience: actually, the coefficient before the inhomogeneous pieces in these transformation laws are defined up to an arbitrary (in general, complex) rescaling of ϕ . The first version is easier to treat by our procedure, so below we just specialize to it.

As in the previously studied cases, it is convenient to deal with the superfields of unbroken supersymmetry, in the present case with $N = (2, 0)$ superfields. The θ^- expansion of the $N = (2, 2)$ chiral superfield ϕ reads

$$\phi(x_L^\pm, x_L^\pm, \theta^+, \theta^-) = (1 - i\theta^- \bar{\theta}^- \partial_-) u(x_L^\pm, x^\pm, \theta^+) + \theta^- \eta^+(x_L^\pm, x^\pm, \theta^+) , \quad (5.7)$$

where u and η^+ are the $N = (2, 0)$ chiral superfields, respectively bosonic and fermionic,

$$\bar{D}_+ u = \bar{D}_+ \eta^+ = 0 . \quad (5.8)$$

The $N = (2, 2)$ superfield transformation (5.4) induces the following ones for these $N = (2, 0)$ superfield components,

$$\delta u = \epsilon^- (\theta^+ + \eta^+) , \quad \delta \eta^+ = -2i\bar{\epsilon}^- \partial_- u . \quad (5.9)$$

Let us now construct the corresponding nonlinear realization by the universal procedure employed in the previous cases.

We perform finite ϵ^- transformations of the superfields η^+ and $D_+ u$, then change, in the transformed superfields, the Grassmann parameters $\epsilon^-, \bar{\epsilon}^-$ to $-\Psi^-, -\bar{\Psi}^-$ where $\Psi^-, \bar{\Psi}^-$ are the Goldstone $N = (2, 0)$ fermionic superfields with the universal ϵ^- transformation law

$$\delta\Psi^- = \epsilon^- - i(\epsilon^- \bar{\Psi}^- + \bar{\epsilon}^- \Psi^-) \partial_- \Psi^- , \quad \delta\bar{\Psi}^- = \bar{\epsilon}^- , \quad (5.10)$$

and, finally, equate to zero the resulting “tilded” superfields with the homogeneous nonlinear transformation law under the ϵ^- transformations. In this way we arrive at the following set of covariant equations

$$\begin{aligned} D_+ u + \Psi^- (1 + D_+ \eta^+) - i\Psi^- \bar{\Psi}^- D_+ \partial_- u &= 0 , \\ \bar{D}_+ \bar{u} + \bar{\Psi}^- (1 - \bar{D}_+ \bar{\eta}^+) + i\Psi^- \bar{\Psi}^- \bar{D}_+ \partial_- u &= 0 , \end{aligned} \quad (5.11)$$

$$\begin{aligned} \eta^+ + 2i\bar{\Psi}^- \partial_- u + i\Psi^- \bar{\Psi}^- \partial_- \eta^+ &= 0 , \\ \bar{\eta}^+ - 2i\Psi^- \partial_- \bar{u} - i\Psi^- \bar{\Psi}^- \partial_- \bar{\eta}^+ &= 0 . \end{aligned} \quad (5.12)$$

As in the previous cases, the first two equations set the equivalence relation between the linear- and nonlinear-realization Goldstone fermion superfields, while the last pair of equations (together with the first one) serves to covariantly express $\eta^+, \bar{\eta}^+$ in terms of the irreducible Goldstone chiral $N = (2, 0)$ superfield u .

These equations can be solved explicitly. As the first step, from (5.11) and (5.12) we find the following relations,

$$\eta^+ = \frac{2i \bar{D}_+ \bar{u} \partial_- u}{1 - \bar{D}_+ \bar{\eta}^+} , \quad \bar{\eta}^+ = -\frac{2i D_+ u \partial_- \bar{u}}{1 + D_+ \eta^+} . \quad (5.13)$$

After some algebra, the relations (5.13) can be rewritten as

$$\begin{aligned}\eta^+ &= 2\bar{D}_+ \left[i\bar{u}\partial_{=u} + \frac{2D_+u\bar{D}_+\bar{u}\partial_{=u}\partial_{=\bar{u}}}{(1+D_+\eta^+)(1-\bar{D}_+\bar{\eta}^+)} \right], \\ \bar{\eta}^+ &= -2D_+ \left[iu\partial_{=\bar{u}} - \frac{2D_+u\bar{D}_+\bar{u}\partial_{=u}\partial_{=\bar{u}}}{(1+D_+\eta^+)(1-\bar{D}_+\bar{\eta}^+)} \right].\end{aligned}\quad (5.14)$$

The numerators of the second terms in the square brackets already contain the maximal number of fermions. Therefore we have to find only the “effective” expressions for the denominators, including no fermions without derivatives. Hitting (5.14) by the corresponding spinor derivatives and discarding all the terms in which fermions appear without x -derivatives on them, one gets the following system of equations:

$$(\bar{D}_+\bar{\eta}^+)_{eff} - \frac{B}{1+(D_+\eta^+)_{eff}} = 0, \quad (D_+\eta^+)_{eff} + \frac{\bar{B}}{1-(\bar{D}_+\bar{\eta}^+)_{eff}} = 0, \quad (5.15)$$

where

$$B = 4\partial_{\mp}u\partial_{=}\bar{u}, \quad \bar{B} = 4\partial_{\mp}\bar{u}\partial_{=}u. \quad (5.16)$$

These algebraic equations have the following solution (with η^+ admitting a power expansion around the point $u = 0, \bar{u} = 0$)

$$\begin{aligned}(D_+\eta^+)_{eff} &= -\frac{1}{2} \left(1 - B + \bar{B} - \sqrt{(1-B-\bar{B})^2 - 4B\bar{B}} \right), \\ (\bar{D}_+\bar{\eta}^+)_{eff} &= \frac{1}{2} \left(1 + B - \bar{B} - \sqrt{(1-B-\bar{B})^2 - 4B\bar{B}} \right).\end{aligned}\quad (5.17)$$

Substituting this into (5.14) we get the final answer for η^+ in terms of u, \bar{u}

$$\eta^+(u, \bar{u}) = 2\bar{D}_+ \left[i\bar{u}\partial_{=u} + \frac{4D_+u\bar{D}_+\bar{u}\partial_{=u}\partial_{=\bar{u}}}{1-B-\bar{B} + \sqrt{(1-B-\bar{B})^2 - 4B\bar{B}}} \right], \quad (5.18)$$

It is straightforward, though tedious, to check that the $\epsilon^-, \bar{\epsilon}^-$ transformations of this $\eta^+(u, \bar{u})$ are in accordance with the law (5.9). Its chirality is manifest in the notation (5.18).

To construct the invariant action, we again first observe that the basic $N = (2, 2)$ superfield ϕ with the transformation law (5.4) admits two invariants (4.19) and (4.20). In terms of the $N = (2, 0)$ superfields defined in (5.7) these invariants take the form

$$S_1 = \frac{1}{f^2} \int d^6z \phi \bar{\phi} = \frac{1}{f^2} \int d^4z^{(+)} [i(u\partial_{=\bar{u}} - \bar{u}\partial_{=u}) + \eta^+\bar{\eta}^+] , \quad (5.19)$$

$$S_2 = \frac{1}{2f^2} \int d^4\zeta_L \phi + \text{c.c.} = -\frac{1}{2f^2} \int d^2x D_+\eta^+ + \text{c.c.} . \quad (5.20)$$

In full analogy with the previous cases, these invariants prove to be identical to each other for the nonlinear realization constructed. Substituting, e.g., the expression (5.18) into S_2 , we obtain

$$S_2(u) = \frac{1}{f^2} \int d^4z^{(+)} \left[i(u\partial_{=\bar{u}} - \bar{u}\partial_{=u}) - \frac{8D_+u\bar{D}_+\bar{u}\partial_{=u}\partial_{=\bar{u}}}{1-B-\bar{B} + \sqrt{(1-B-\bar{B})^2 - 4B\bar{B}}} \right]. \quad (5.21)$$

Using eqs (5.13) - (5.18), it is easy to show that

$$S_2(u) = S_1(u) . \quad (5.22)$$

The bosonic part of the action (5.21) reads

$$S^{bos}(u) = \frac{1}{2f^2} \int d^2x \left\{ 1 - \sqrt{(1 - B - \bar{B})^2 - 4B\bar{B}} \right\} \quad (5.23)$$

which is the static-gauge form of the Nambu-Goto action of the string in $D = 4$ Minkowski space, the real and imaginary parts of $u(x)$ being two transversal coordinates. The authors of [8] have explicitly shown that (5.21) is the manifestly world-sheet supersymmetric form of the Green-Schwarz action of an $N = 1$ superstring in $D = 4$ (they used a different parametrization for the Goldstone multiplet, representing it by a superfield subject to some nonlinear chirality constraints, in contrast to the ordinary chiral Goldstone superfield u in our approach). This form arises after fixing local fermionic κ -symmetry so as to kill half of the target spinor coordinates. It is straightforward to see that the algebra (2.17) with $Z_1 = Z_2 = 0$ coincides with the $N = 1, D = 4$ Poincaré superalgebra after combining the $D = 2$ spinor generators into $D = 4$ Weyl spinors via

$$(\bar{Q}_+, Q_-) \equiv Q_\alpha, \quad (Q_+, \bar{Q}_-) \equiv \bar{Q}_{\dot{\alpha}}$$

and identifying P_+, P_- and $Z_3 + iZ_4$ with the components of the 4-momentum generator. Note that the action (5.21) can be also recovered by world-volume dimensional reduction from the $D = 4$ chiral Goldstone superfield action of [10] which corresponds to the $N = 1$ super 3-brane in $D = 6$.

We conclude by a few comments.

First, in the case under consideration, without using derivatives, it is impossible to construct a homogeneously transforming superfield like (4.12) because the central charge acts as a complex shift of ϕ . Hence the above nonlinear realization, as opposed to the one considered in the previous Section, cannot be reproduced alternatively by imposing a nilpotency condition on some proper superfield along the lines of ref. [12, 13]. At the same time, our general procedure works pretty well in this case, too. It definitely amounts to imposing some covariant constraints on the linear-realization Goldstone superfield ϕ , but they necessarily include derivatives and it is rather hard to immediately guess their precise form.

Second, the Goldstone chiral $N = (2, 0)$ superfield u collects only physical degrees of freedom (transverse string coordinates, or Goldstone fields for the complex central charge, and the complex Goldstone fermion). So the case under consideration cannot be related to a D1-brane, in contrast to the option considered in Sect. 4. It seems also impossible to make use of the vector $N = (2, 0)$ multiplet as an alternative Goldstone one to represent the partial breaking pattern considered here. Indeed, this multiplet can be described by the chiral fermionic superfield strength Ξ_v^+ , eq. (3.46). It has the same $SO(1, 1)$ weight as θ^+ and for this reason cannot be utilized as a Goldstone fermion for the partial breaking $N = (2, 2) \rightarrow N = (2, 0)$ (such a fermion should have the same weight as θ^-). On the other hand, it could support the 1/2 breaking $N = (4, 0) \rightarrow N = (2, 0)$. A difference from the case $N = (2, 0) \rightarrow N = (1, 0)$ with the vector Goldstone $N = (1, 0)$ multiplet (Sect. 3.2) is that the $N = (2, 0)$ vector multiplet contains a scalar auxiliary field (in addition to the complex chiral fermion and $D = 2$ gauge field strength). So the corresponding Goldstone superfield action is expected to be less trivial.

Third, let us note that the initial linear realization (5.2) (with $Z_1 + iZ_2 = 0$) is equivalent, by the Z -frame rotation with $A = 2\theta^-\theta^+$, to the following one

$$\begin{aligned} \hat{Q}_+ &= Q_+ - 2\theta^-(Z_3 + iZ_4), & \hat{\bar{Q}}_+ &= \bar{Q}_+ \\ \hat{Q}_- &= Q_-, & \hat{\bar{Q}}_- &= \bar{Q}_-. \end{aligned} \quad (5.24)$$

This choice corresponds to the PBGS pattern $N = (2, 2) \rightarrow N = (0, 2)$. The relevant nonlinear-realization Goldstone multiplet is represented by a chiral $N = (0, 2)$ superfield $u(x^\pm, x_L^\pm, \theta^-)$. The Goldstone superfield action is related to (5.21) by a mirror symmetry and amounts to an alternative gauge-fixing of κ -symmetry in the $N = 1, D = 4$ superstring Green-Schwarz action.

6 Partial breaking $N=(2,2)$ to $N=(1,0)$

As was noticed in Sect.4, before imposing any restrictions on the central charges in (4.2) (apart from the condition (2.26) required for the existence of chiral $N = (2, 2)$ superfields), one can define a chiral superfield with the transformation law corresponding to 1/4 partial breaking of $N = (2, 2)$ supersymmetry, i.e., such that only one real supersymmetry acts by homogeneous transformations. By means of the appropriate Z -frame rotation, the generators (4.2) can be brought into the form

$$\begin{aligned}\hat{Q}_+ &= Q_+ - 2\theta^-(Z_1 - iZ_2) , \\ \hat{\bar{Q}}_+ &= \bar{Q}_+ + 2\theta^-(Z_1 - iZ_2) , \\ \hat{Q}_- &= Q_- - 2\theta^+[(Z_3 + iZ_4) - (Z_1 - iZ_2)] , \\ \hat{\bar{Q}}_- &= \bar{Q}_- + 2\theta^+(Z_1 + iZ_2) .\end{aligned}\tag{6.1}$$

With all the central charge generators realized as complex shifts of the chiral superfield ϕ , it follows from the general transformation law,

$$\delta\phi = (\epsilon\hat{Q})\phi ,\tag{6.2}$$

that only the ϵ_1^+ supersymmetry generated by $\hat{Q}_+^1 \sim \hat{Q}_+ + \hat{\bar{Q}}_+$ is unbroken, because only its transformation contains no central charge terms.

Like in the previous examples, there exist quite a few possibilities to choose a particular realization of central charges with preserving the above crucial property. In this paper we do not aim at the exhaustive analysis of all possibilities, and choose this realization so as to make the computations most feasible,

$$(a) (Z_1 + iZ_2)\phi = 0 , \quad (b) (Z_1 - iZ_2)\phi = \frac{i}{2} , \quad (c) (Z_3 + iZ_4)\phi = -(Z_1 - iZ_2)\phi = -\frac{i}{2} .\tag{6.3}$$

With this ansatz, the transformation law (6.2) becomes

$$\delta\phi = -2i\theta^+\epsilon^- - \sqrt{2}\theta^-\epsilon_2^+ + (\epsilon Q)\phi .\tag{6.4}$$

Let us now consider the $N = (1, 0)$ decomposition of the $N = (2, 2)$ chiral superfield

$$\phi = (1 + i\theta_2^+ D_+^1)[(1 - \theta_1^- \theta_2^- \partial_-)(\phi_1 + i\phi_2)(x, \theta_1^+) + (\theta_1^- + i\theta_2^-)(\pi^+ + i\eta^+)(x, \theta_1^+)] ,\tag{6.5}$$

where ϕ_1 , ϕ_2 and π^+ , η^+ are the real bosonic and fermionic $N = (1, 0)$ superfields. The $N = (2, 2)$ superfield transformation law (6.4) implies the following transformation laws for these $N = (1, 0)$ projections

$$\begin{aligned}\delta\phi_1 &= i\epsilon_1^- (\theta^+ + \eta^+) - \epsilon_2^+ D_+ \phi_2 + i\epsilon_2^- \pi^+ , \\ \delta\phi_2 &= i\epsilon_2^- (\theta^+ + \eta^+) + \epsilon_2^+ D_+ \phi_1 - i\epsilon_1^- \pi^+ , \\ \delta\pi^+ &= -\epsilon_2^+ (1 + D_+ \eta^+) + \epsilon_1^- \partial_- \phi_2 - \epsilon_2^- \partial_- \phi_1 , \\ \delta\eta^+ &= \epsilon_2^+ D_+ \pi^+ - \epsilon_1^- \partial_- \phi_1 - \epsilon_2^- \partial_- \phi_2\end{aligned}\tag{6.6}$$

(hereafter we omit the index 1 of the real $N = (1, 0)$ superspace Grassmann coordinate, $\theta_1^+ \rightarrow \theta^+$). As is seen from (6.6), the spinor superfield π^+ is the Goldstone fermion for the $N = (0, 1)$ supersymmetry with the parameter ϵ_2^+ . Two other Goldstone fermions which are required by partial breaking of the remaining $N = (0, 2)$ supersymmetry are the spinor derivatives of the superfields ϕ_1 and ϕ_2 :

$$\begin{aligned}\xi_+^1 \equiv iD_+ \phi_1 &\Rightarrow \delta\xi_+^1 = \epsilon_1^- (1 + D_+ \eta^+) + \epsilon_2^+ D_+ \xi_+^2 + \epsilon_2^- D_+ \pi^+ , \\ \xi_+^2 \equiv iD_+ \phi_2 &\Rightarrow \delta\xi_+^2 = \epsilon_2^- (1 + D_+ \eta^+) - \epsilon_2^+ D_+ \xi_+^1 - \epsilon_1^- D_+ \pi^+ .\end{aligned}\tag{6.7}$$

The superfields ϕ_1 and ϕ_2 are the Goldstone ones for two independent central-charge shifts of $\phi(\zeta_L)$.

The superfield η^+ transforms homogeneously with respect to all three spontaneously broken supersymmetries. In virtue of the transformation properties (6.6) one can construct the simplest off-shell invariant

$$S = -i \int d^3 z^{(+)} \eta^+ = \int d^2 x D_+ \eta^+ \quad (6.8)$$

(hereafter, for convenience, we put normalization factors before actions equal to 1). It is the $N = (1, 0)$ superfield form of the universal action S_2 defined in (4.20), which, along with another such action, S_1 , eq. (4.19), are invariant under the realization (6.4), similarly to the cases of other inhomogeneous realizations of $N = (2, 2)$ supersymmetry considered in the previous Sections. The precise form of these two invariants is as follows

$$S_1 \sim \frac{1}{4} \int d^6 z \phi \bar{\phi} = \int d^3 z^{(+)} \left[\sum_{i=1,2} \partial_{=} \phi_i D_+ \phi_i - i (\pi^+ D_+ \pi^+ + \eta^+ D_+ \eta^+) \right], \quad (6.9)$$

$$S_2 \sim \frac{i}{4} \int d^4 \zeta_L \phi + \text{c.c.} = -i \int d^3 z^{(+)} \eta^+. \quad (6.10)$$

Like in the previous cases, one possibility to make the invariant (6.8) meaningful is to covariantly express the superfield η^+ in terms of Goldstone superfields $\xi_+^{1,2}, \pi^+$. In other words, we have to construct from Goldstone superfields the spinor superfield which transforms as η^+ with respect to the full $N = (2, 2)$ supersymmetry. The idea of such a construction is the same as before: we write the finite supersymmetric transformation of our superfields $\xi_+^{1,2}, \pi^+, \eta^+$ and replace the parameters $\{-\epsilon_2^+, -\epsilon_{1,2}^+\}$ by the Goldstone fermionic superfields of nonlinear realizations $\{\psi_2^+, \psi_{1,2}^+\}$:

$$\begin{aligned} \tilde{\xi}_+^1 &= \xi_+^1 - \psi_2^+ D_+ \xi_+^2 - \psi_1^- (1 + D_+ \eta^+) - \psi_2^- D_+ \pi^+ \dots, \\ \tilde{\xi}_+^2 &= \xi_+^2 + \psi_2^+ D_+ \xi_+^1 + \psi_1^- D_+ \pi^+ - \psi_2^- (1 + D_+ \eta^+) + \dots, \\ \tilde{\pi}^+ &= \pi^+ + \psi_2^+ (1 + D_+ \eta^+) - \psi_1^- \partial_{=} \phi_2 + \psi_2^- \partial_{=} \phi_1 + \dots, \\ \tilde{\eta}^+ &= \eta^+ - \psi_2^+ D_+ \pi^+ + \psi_1^- \partial_{=} \phi_1 + \psi_2^- \partial_{=} \phi_2 + \dots. \end{aligned} \quad (6.11)$$

Here we have explicitly written down only linear terms, denoting the higher-order terms by dots. The transformation properties of the universal Goldstone fermionic superfields are analogous to those given earlier, so we do not explicitly quote them.

Once again, one can check that the new superfields $\tilde{\xi}_+^{1,2}, \tilde{\pi}^+, \tilde{\eta}^+$ transform homogeneously and hence can be covariantly equated to zero. These conditions provide the sought system of equations relating Goldstone fermions of the linear $\{\xi_+^{1,2}, \pi^+\}$ and nonlinear $\{\psi_2^+, \psi_{1,2}^+\}$ realizations and also serving to covariantly express the superfield η^+ in terms of the remaining superfields.

The system (6.11) is much more complicated than those we encountered in the previous Sections. It can be solved by iterations, but the solution looks not too enlightening. Let us concentrate on the pure bosonic part of the action which we firstly choose to coincide with the simplest invariant (6.8). In the bosonic limit we can keep in eqs. (6.11) only the terms written explicitly, because the rest is at least of the third order in fermions. Therefore, in this approximation our systems can be written as

$$\begin{pmatrix} \pi^+ \\ \xi_+^1 \\ \xi_+^2 \end{pmatrix} - A \begin{pmatrix} -\psi_2^+ \\ \psi_1^+ \\ \psi_2^- \end{pmatrix} = 0, \quad (6.12)$$

$$\eta^+ - \psi_2^+ D_+ \pi^+ + \psi_1^- \partial_{=} \phi_1 + \psi_2^- \partial_{=} \phi_2 = 0, \quad (6.13)$$

where

$$A = \begin{pmatrix} 1 + D_+\eta^+ & \partial_=\phi_2 & -\partial_=\phi_1 \\ -D_+\xi_+^2 & 1 + D_+\eta^+ & D_+\pi^+ \\ D_+\xi_+^1 & -D_+\pi^+ & 1 + D_+\eta^+ \end{pmatrix}. \quad (6.14)$$

Now we can solve the equation (6.12) for $\{\psi_2^+, \psi_{1,2}^-\}$ and substitute the solution into (6.11). After hitting (6.13) with one spinor derivative we finally obtain the equation

$$D_+\eta^+ = - (D_+\pi^+, \partial_=\phi_1, \partial_=\phi_2) A^{-1} \begin{pmatrix} D_+\pi^+ \\ D_+\xi_+^1 \\ D_+\xi_+^2 \end{pmatrix}. \quad (6.15)$$

Let us recall that $a \equiv D_+\eta^+$ is the bosonic Lagrangian density we are looking for (see eq. (6.8)). Thus the bosonic Lagrangian can be found as a solution of eq. (6.15) which amounts to the following quartic equation:

$$(a^2 + 2a + 1 + y)(a^2 + a + y) + z(z + k) = 0, \quad (6.16)$$

where

$$\begin{aligned} k &\equiv D_+\pi^+, \quad y = k^2 + \partial_=\phi_1 \partial_{\neq} \phi_1 + \partial_=\phi_2 \partial_{\neq} \phi_2, \\ z &= \partial_=\phi_1 \partial_{\neq} \phi_2 - \partial_=\phi_2 \partial_{\neq} \phi_1. \end{aligned} \quad (6.17)$$

To find the solution of eq.(6.16) to low orders in the fields, we firstly employ in (6.16) the equation of motion for the auxiliary field k :

$$\frac{\partial}{\partial k} a = 0 \Rightarrow 2k(2a^2 + 3a + 2y + 1) + z = 0. \quad (6.18)$$

The solution of the system (6.16) plus (6.18), up to eighth order in physical fields, reads

$$a = -\tilde{y} - \frac{3}{4}z^2 - \tilde{y}^2 - 2\tilde{y}z^2 - 2\tilde{y}^3 - \frac{19}{16}z^4 - \frac{25}{4}\tilde{y}^2z^2 - 5\tilde{y}^4 + \dots, \quad (6.19)$$

$$k = -\frac{z}{2} \left(1 + \tilde{y} + \frac{7}{4}z^2 + 2\tilde{y}^2 + \frac{11}{2}\tilde{y}z^2 + 5\tilde{y}^3 \right) + \dots, \quad (6.20)$$

where

$$\tilde{y} = \partial_=\phi_1 \partial_{\neq} \phi_1 + \partial_=\phi_2 \partial_{\neq} \phi_2.$$

One can find a general solution of eq. (6.16), but even after eliminating the auxiliary field $k = D_+\pi^+$ this solution does not look very informative.

To find a concise expression for the bosonic action, let us recall that, besides the Lagrangian $L_1 = a$, we have another one, which is equal to the free Lagrangian of our original linear $N = (2, 2)$ supermultiplet

$$L_{free} = a^2 + y \quad (6.21)$$

(this expression is obtained from (6.9) after integrating over θ^+ and discarding the fermions). Let us define a new bosonic Lagrangian as some combination of L_{free} and L_1 (up to an overall normalization factor),

$$L = L_{free} + \alpha L_1 \equiv a^2 + \alpha a + y, \quad (6.22)$$

where α is an arbitrary parameter for the time being.

The field k is auxiliary, and therefore it can be eliminated by solving its algebraic equation of motion:

$$\frac{\partial}{\partial k} L \equiv (2a + \alpha) \frac{\partial}{\partial k} a + 2k = 0 \Rightarrow \frac{\partial}{\partial k} a = -\frac{2k}{2a + \alpha}. \quad (6.23)$$

Differentiating (6.16) with respect to k and taking account of (6.23), we get the equation

$$-2k[(2-\alpha)(L+(1-\alpha)a)+(1-\alpha)(L+(2-\alpha)a+1)]+(2a+\alpha)z=0. \quad (6.24)$$

Now, fixing our free parameter α to

$$\alpha=2, \quad (6.25)$$

we find the following unique solution of eqs. (6.16) and (6.24),

$$k=-z, \quad a^2+a+\tilde{y}=0, \quad (6.26)$$

where

$$\tilde{y}=z^2+\left(\partial_{\mp}v\partial_{\mp}v+\partial_{\mp}s\partial_{\mp}s\right)|_{\theta=0}. \quad (6.27)$$

Therefore,

$$a=-\frac{1}{2}\left(1-\sqrt{1-4\tilde{y}}\right), \quad (6.28)$$

and (6.22) becomes

$$L=-\frac{1}{2}\left(1-\sqrt{1-4\tilde{y}}\right)=a. \quad (6.29)$$

The physical Lagrangian $L_{phys}=-L$ describes a bosonic string in $D=4$ Minkowski space. The full Goldstone-superfield Lagrangian still drastically differs in fermionic terms from that of the standard $N=1, D=4$ superstring of Sect. 5 (three physical Goldstone fermions are present now as compared to two such fermions of the previous example).

We end with a few comments.

First, recall that with the simultaneous presence of two sorts of central charges, Z_1-iZ_2 and Z_3+iZ_4 , in the $N=(2,2)$ superalgebra (2.17), the latter is a reduction of the tensor-central-charge-extended $N=1, D=4$ superalgebra [22]-[24]. Since both sorts of central charges are present in the transformation law (6.4) or (6.3), from the $N=1, D=4$ Poincaré superalgebra standpoint the transversal coordinates ϕ_1 and ϕ_2 of the string are associated with some combinations of the standard four-momentum and the tensorial central charges. In other words, the ambient bosonic manifold of our system is non-trivially embedded into the product of ordinary Minkowski space and the space parametrized by coordinates conjugate to the tensorial central charges. A similar situation was observed in the superparticle models with 1/4 breaking [21]. A recent paper [33] also discusses the possibility to associate transverse brane coordinates with components of tensorial central charges, rather than with those of the conventional momentum operator. If one repeats for the present case the model-independent analysis undertaken in ref. [24] to classify admissible BPS configurations in $N=1, D=4$ supersymmetry with tensorial central charges, one will find that a choice of central charges as in (6.3) allows indeed just for the 1/4 breaking option. Yet, it is still unclear to us how the essentially on-shell analysis of [24] correlates with our approach which proceeds from off-shell superfield representations of $N=(2,2)$ supersymmetry.

Second, the fractional patterns of PBGS other than 1/2, in particular, the 1/4 one, are known to naturally occur in systems of intersecting branes (see, e.g., [34]). It would be interesting to elaborate on a similar interpretation for our string-like 1/4 system. A closely related problem is to construct an appropriate Green-Schwarz-type action. It should respect only one real κ -supersymmetry, break $D=4$ Lorentz invariance and, in the static gauge, reproduce the on-shell form of our Goldstone superfield action (still to be fully worked out).

Third, like in the examples of Subsect.3.2 and Sect.4, one could pass to a “1/4 super D1-brane” by substituting the covariant field strength of the $D=2$ Maxwell field for the auxiliary field $k=D_+\pi^+$ in the above relations. As the auxiliary field now plays an important role in forming the correct string-type bosonic action, one expects this trick to have a greater impact on the structure of the relevant actions than compared to the previous cases.

7 Conclusions

In this paper we have constructed universal low-energy nonlinear Goldstone superfield actions for a few patterns of partial breaking of $N = (1, 1)$, $N = (2, 0)$ and $N = (2, 2)$ supersymmetries in two dimensions. We have shown that these actions provide a manifestly world-sheet supersymmetric description of some superstrings and D1-branes in flat $D = 3$ and $D = 4$ Minkowski backgrounds. One novelty of our treatment is that we proceeded from a purely two-dimensional setting, without assuming embeddings into higher dimensions in advance. We found that in most cases, for the partial breaking to occur, the supersymmetries must necessarily be extended by appropriate central-charge generators. The latter produce pure shifts of scalar fields in the Goldstone multiplets, allowing one to identify these fields with the transversal coordinates of some brane in the static gauge.

As another novel point our construction systematically exploits the general relation between linear and nonlinear realizations of spontaneously broken supersymmetries [20]. This allows us to deduce new equivalent forms of the Goldstone superfield actions in terms of the superfields of the full supersymmetry. These superfields transform linearly (though inhomogeneously) under the broken part of the supersymmetry and are nonlinear functions of the basic irreducible Goldstone multiplets. In such a representation both kinds of supersymmetries, broken as well as unbroken, are manifest. Besides the $1/2$ partial breaking options, we considered the $1/4$ breaking of $N = (2, 2)$ supersymmetry, showed the existence of a manifestly supersymmetric Goldstone superfield action for this PBGS pattern and found its bosonic piece. It would be of great interest to find the corresponding Green-Schwarz-type action (if existing).

We did not aim to give an exhaustive analysis of all possible schemes and realizations of partial breaking of two-dimensional supersymmetries. Our goal was to construct low-energy Goldstone superfield actions for a few technically feasible cases. For the convenience of the reader, we summarize our basic examples in the Table below. We quote the superfields of linear realization (LR), the minimal Goldstone multiplets (GM), the non-vanishing central charges (CC), and the space-time interpretation of the Goldstone superfield actions.

| PBGS | LR | GM | CC | Interpretation |
|-----------------|-----------------------------------|---|------------------------|----------------------|
| $(1, 1)/(1, 0)$ | $\Phi(x, \theta_1^\pm)$ | $u(x, \theta_1^+)$ | Z | $N = 1, D = 3$ sstr. |
| $(2, 0)/(1, 0)$ | $\Xi^+(x^=, x_L^\pm, \theta^+)$ | $\xi^+(x, \theta_1^+)$ | No | left-chir. ferm. |
| | $\Xi_v^+(x^=, x_L^\pm, \theta^+)$ | $\xi_v^+(x, \theta_1^+)$ | No | space-filling D1-br. |
| $(2, 2)/(1, 1)$ | $\phi(\zeta_L)$ | $\pi(x, \theta_1^\pm)$ | Z_2 | $N = 2, D = 3$ sstr. |
| | $\phi_v(\zeta_L)$ | $\pi_v(x, \theta_1^\pm)$ | Z_2 | $D = 3$ D1-br. |
| $(2, 2)/(2, 0)$ | $\phi(\zeta_L)$ | $u(x^=, x_L^\pm, \theta^+)$ | Z_3, Z_4 | $N = 1, D = 4$ sstr. |
| $(2, 2)/(1, 0)$ | $\phi(\zeta_L)$ | $\phi_{1,2}(x, \theta_1^+), \pi^+(x, \theta_1^+)$ | $Z_1 - Z_3, Z_2 - Z_4$ | ? |

We finish with a few comments and proposals for future study.

When setting up linear realizations of the partially broken supersymmetries, we proceeded from the simplest supermultiplets of the latter. In particular, all patterns of the $N = (2, 2)$ supersymmetry breaking were realized on a single $N = (2, 2)$ chiral superfield, while the different PBGS options were realized by choosing different sets of central charges in the $N = (2, 2)$ superalgebra. We could equally reproduce all these options, with the same final minimal Goldstone superfield actions, by starting from a twisted-chiral $N = (2, 2)$ superfield. In this case, the manifestly $N = (2, 2)$ supersymmetric form of these actions is given by expressions like (4.19) and (4.20), with twisted-chiral superfields instead of the chiral ones. These two equivalent representations of the same Goldstone superfield action are related to each other by mirror symmetry (see, e.g., [6]) which interchanges

chiral and twisted-chiral superfields and amounts to the twists $\theta^+ \leftrightarrow \bar{\theta}^+$ or $\theta^- \leftrightarrow \bar{\theta}^-$ accompanied by appropriate reflections of the irreducible Goldstone superfields. One more possibility is to start from a semi-chiral $N = 2$ superfield [35]. Once again, based on the universality of the Goldstone superfield actions, we expect the corresponding linear realizations of the $N = (2, 2)$ PBGS patterns to lead, upon applying our general procedure, to the same nonlinear realizations and minimal Goldstone superfield actions modulo field redefinitions.

New opportunities arise when more than one $N = (2, 2)$ superfield is incorporated simultaneously. For instance, a chiral plus a twisted chiral superfield form a simplest off-shell representation of $N = (4, 4)$ supersymmetry, the twisted chiral $N = (4, 4)$ multiplet [1, 36]. Thus one can implement various versions of the partial breaking of $N = (4, 4)$ supersymmetry using this simple system. This extended $D = 2$ supersymmetry is related, via dimensional reduction, to $N = 4$ $D = 3$, $N = 2$ $D = 4$, $N = 1$ $D = 5$ and $N = 1$ $D = 6$ supersymmetries. The relevant $D = 2$ Goldstone superfield actions are expected to describe superstrings and D1-branes associated with these supersymmetries and spacetimes. Furthermore, it is of interest to construct similar PBGS models for heterotic $N = (4, 0)$ supersymmetry. In both cases, besides the $N = (2, 2)$ and $N = (2, 0)$ superfield descriptions, there exists a concise off-shell description in terms of $D = 2$ harmonic superfields [37, 38] manifesting all supersymmetries. Hence, it is tempting to generalize our construction to harmonic $N = (4, 4)$ and $N = (4, 0)$ superfields. A natural further step is to try to construct the $D = 2$ PBGS actions with partially broken $N = (8, 8)$ and $N = (16, 16)$ supersymmetries (and their heterotic truncations). They should be relevant to $D = 10$ superstrings.

In the above consideration we systematically identified central charges with the generators of shift isometries of the superfields involved. New possibilities for model-building could come out in mixed cases, when some central charges are set to generate homogeneous rotational isometries.

Another line of future study may consist in generalizing the Goldstone superfield actions presented in this paper, by adding to them covariant couplings of Goldstone superfields to other matter and gauge superfields and by coupling them to the appropriate world-sheet supergravities. Such extended PBGS models should amount to a static-gauge formulation of superstrings and D1-branes evolving in non-trivial curved backgrounds.

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